Universal Turing Machines and Diagonalization

Mark Greenstreet, CpSc 421, Term 1, 2008/09

- Universal Turing Machines
  - A Turing Machine that can be programmed to simulate any other Turing Machine.
- Diagonalization
  - A way to show compare the sizes of infinite sets.
  - On Wednesday, we'll use it to give a formal proof that the Halting Problem is undecidable.

Some "Universal" Languages

- $A_R = \{D\#w \mid D$ describes a DFA that accepts string $w\}$
  - This is the "universal" language for Regular Languages.
  - We described a Turing Machine for A_R in the Oct. 24 lecture.
- $A_{CFL} = \{G\#w \mid G$ describes a CFG that generates string $w\}$
  - This is the "universal" language for Context-Free Languages.
  - We described a Turing Machine for A_{CFL} in the Oct. 24 lecture.
- $A_{TM} = \{M\#w \mid M$ describes a TM that accepts string $w\}$
  - This is the "universal" language for Turing Recognizable Languages.
  - We'll described a Turing Machine for A_{TM} now.

A Universal Turing Machine

$A_{TM} = \{M\#w \mid M \text{ describes a TM that accepts string } w\}$

We'll define a Turing Machine, U, that recognizes A_{TM}.

Σ_U: $\{\texttt{0}, \texttt{1}, \texttt{(}, \texttt{,}, \texttt{)}, \texttt{\#}\}$

Γ_U: $\Sigma \cup \{\square, \ldots\}$

w: The format for the input tape is described on the next slide.

Tapes: We'll use six tapes:

$input$	$=$	The input string, $M\#w$ is written here.
δ_M	$=$	A list of tuples representing the transition function of M is written here.
q_M	$=$	The current state of M is written here.
c_M	$=$	The current tape symbol of M is written here.
$tape_M$	$=$	The current tape contents for M.
$scratch$	$=$	A scratch tape.

Input Tape Format for U

$|Q_M|$, $|\Sigma_M|$, $|\Gamma_M|$, δ_M#w where

- $|Q_M|$: Binary representation of the number of states of M.
- $|\Sigma_M|$: Binary representation of the number of symbols in the input alphabet of M.
- $|\Gamma_M|$: Binary representation of the number of symbols in the tape alphabet of M.
- δ_M: A list of tuples for the transition function for M. Each tuple has the form: (q, c, q', c', d) where $\delta_M(q, c) = (q', c', d)$. In other words, when M is in state q and reads c, it transitions to state q', writes a c' on the tape and moves one square in direction d, $d \in \{$0, 1$\}$, where 0 denotes a left move and 1 denotes a right move.
- q_0, $accept$, and $reject$: we assume that these special states are represented by 0, 1, and 2 respectively.
- w: The input string: binary numbers separated by commas. We assume that each symbol in Γ is encoded using the same number of bits, $\lceil \log_2 |\Gamma| \rceil$.

Operation of U (1/2)

Make sure the input is valid:

- Check that the tape has the form $B^*\texttt{,}B^*\texttt{,}B^*\texttt{,}C^*\texttt{\#}B^*(\texttt{,}B^*)^*$ where

  $B = \{\texttt{0}, \texttt{1}\}$

  $C = \texttt{(}B^*\texttt{,}B^*\texttt{,}B^*\texttt{,}B^*\texttt{,}B^*\texttt{)}$

  Note: This format requirement is a regular language. U can check this by scanning the tape from left-to-right using its finite states to implement a DFA.

- Read $|Q_D|$, $|\Sigma_D|$ and $|\Gamma_D|$ and copy them onto the appropriate tapes.
- Copy δ_M onto the δ_M tape.
- Make sure that each tuple, (q, c, q', c', d) for δ_M has $q, q' \in 0 \ldots (|Q_D| - 1)$, $c, c' \in 0 \ldots (|\Gamma_D| - 1)$, $d, \in B$. Make sure all combinations for q and c are covered.
- Copy w onto the $tape_M$ tape —
  write the binary string for M's blank if $w = \epsilon$.
- Make sure that each symbol for w is in Σ_D.

Operation of U (2/2)

- Simulate M.

```
q ← 0
while(q ∉ {1,2}) {
  c ← string in B* under head on tape_M.
    (if there is a blank under the head,
      write a comma and the binary string
      for M's blank)
  scan δ_M tape to find entry for (q,c),
    let this be (q,c,q',c',d)
  copy q' onto the q tape.
  copy c' onto the tape_M tape.
  move head for tape_M according to d.
}
if(q == 1) accept;
else reject.
```

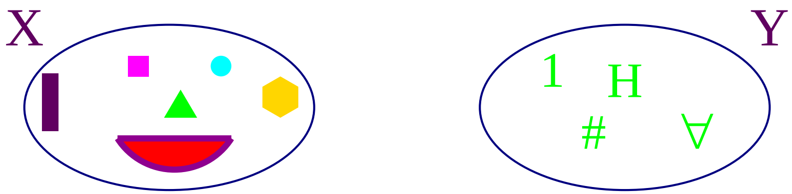
Observations

- If M accepts w, then U accepts $M\#w$.
- If M rejects w, then U rejects $M\#w$.
- If M loops on w, then U loops on $M\#w$.
- $\therefore U$ recognizes A_{TM}.
- U is universal:
  - One machine U works with any input $M\#w$.
    In other words, U can simulate any Turing machine, M.
  - You can think of the M part of $M\#w$ as a program, and the w part as the input data for the program.
  - U is a programmable machine. Rather than building a new TM for each problem, we just program U to simulate whatever TM we want.

Halting for Turing Machines

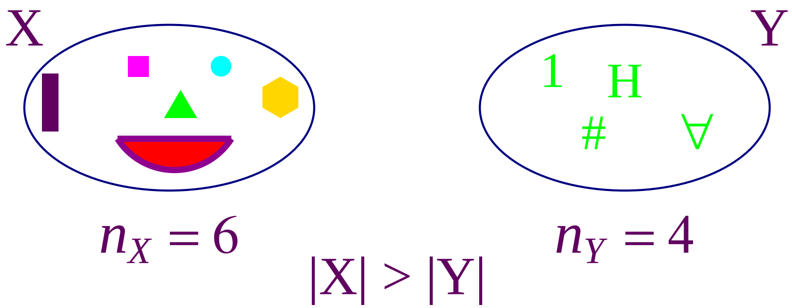
- From the previous slide, U loops on input $M\#w$ iff M loops on input w.
- We've shown that U recognizes A_{TM}, but it doesn't decide A_{TM}.
- Could we build some other machine, U' that can determine when a machine M loops on its given input? If so, then U' would decide A_{TM}.
  - This would require solving the Halting Problem for Turing Machines.begin
  - We gave an informal argument (see the Oct. 24 slides) that the Halting Problem for Java^{TM} programs is undecidable (by Java programs). On Wednesday (Oct. 29), we'll show that the Halting Problem for Turing Machines is undecidable.
  - First, we'll look at "diagonalization", the main mathematical concept that we'll need for the proof.

Which Set is Bigger?



- Let X and Y be sets.
- Is $|X| > |Y|$?
- Solution by counting:
  - Count each element in X. Let n_X be the number.
  - Count each element in Y. Let n_Y be the number.
  - If $n_X > n_Y$, then $|X| > |Y|$.

Which Set is Bigger?



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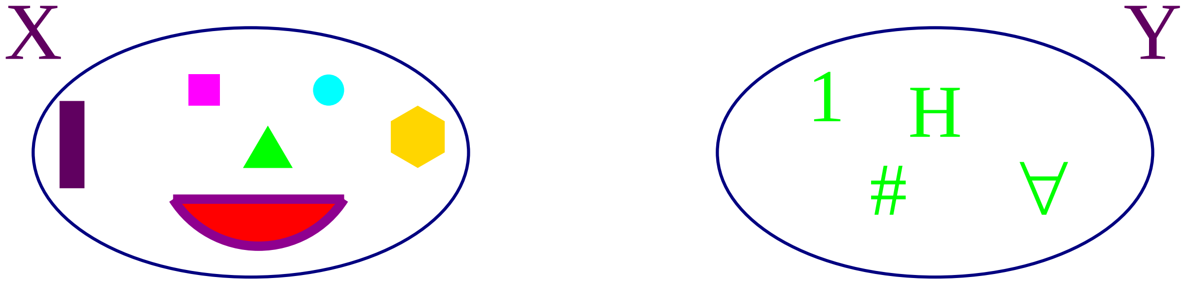
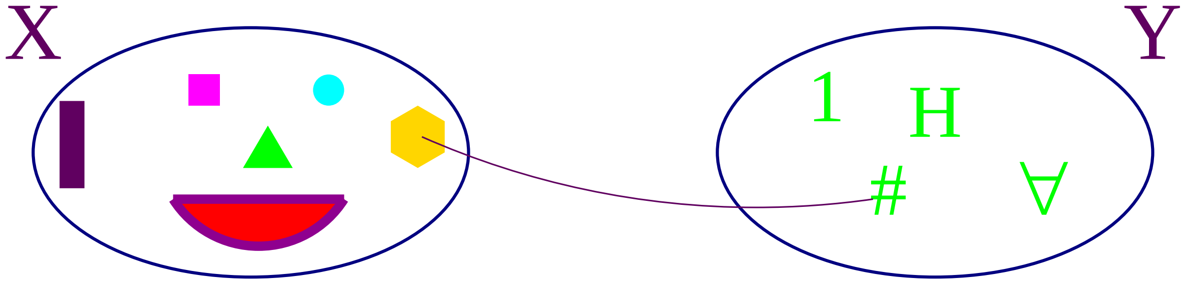
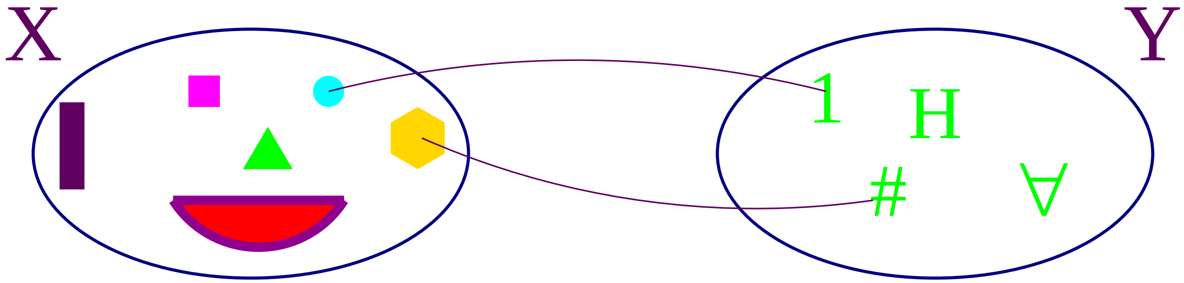
Comparing by Pairing



Comparing by Pairing



Comparing by Pairing



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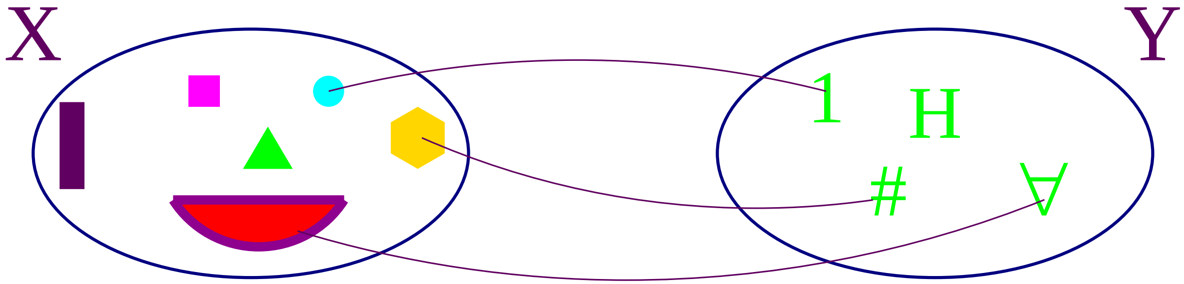

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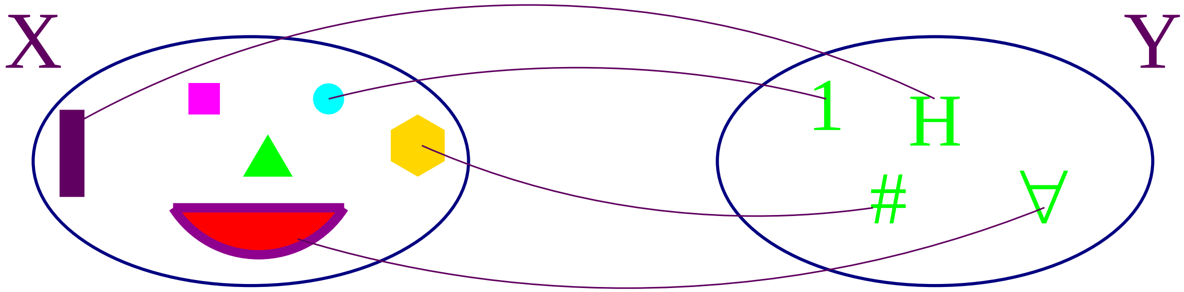
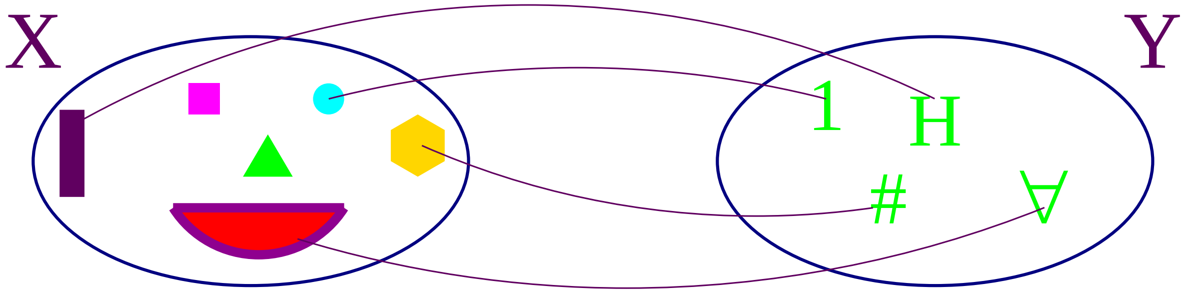


Comparing by Pairing



$$|X| > |Y|$$

- If there is an onto function, $f : X \to Y$, then $|X| \geq |Y|$.
- If there are onto functions $f : X \to Y$ and $g : Y \to X$, then $|X| \geq |Y|$ and $|Y| \geq |X|$, Thus, $|X| = |Y|$.
- Note that if $f : X \to Y$ is one-to-one and onto, then f^{-1} exists and is one-to-one, and onto as well. Thus, if there is a one-to-one and onto function, $f : X \to Y$, then $|X| = |Y|$.
- If there is no onto function $g : Y \to X$, then $|X| > |Y|$.

Even Integers vs. All Integers

- Let $\mathbb{Z}$ be the set of all integers, and $\mathbb{E}$ be the set of all even integers.
  - Let $f : \mathbb{Z} \rightarrow \mathbb{E}$ be the function $f(x) = 2x$.
  - f is one-to-one: If $f(x) = f(y)$, then $2x = 2y$, and $x = y$.
  - f is onto: If $y \in \mathbb{E}$, then $y/2 \in \mathbb{Z}$, and $f(y/2) = y$.
  - $\therefore \mathbb{E} = \mathbb{Z}$.

    In English, this says that the number of even integers is equal to the number of all integers!
- A similar argument shows that $|\mathbb{N}| = |\mathbb{Z}|$.

Naturals vs. Rationals (1/2)

- Let $\mathbb{Q}^+$ be the set of all strictly-positive rational numbers, and $\mathbb{N}^+$ be the strictly-positive naturals.
- Let $f : \mathbb{Q}^+ \rightarrow \mathbb{N}^+$ with $f(x) = \lceil x \rceil$. Clearly, f is onto, thus $|\mathbb{Q}^+| \geq |\mathbb{N}^+|$ — there are at least as many positive rational numbers as positive naturals.
- Let $g : \mathbb{N}^+ \rightarrow \mathbb{Q}^+$ with
$$\begin{aligned} g(n) &= \tfrac{x(n)+1-z(n)}{z(n)} \\ x(n) &= \left\lfloor \tfrac{1}{2}(\sqrt{8n-7}+1) \right\rfloor \\ y(n) &= \tfrac{1}{2}(x(n)^2 - x(n)) \\ z(n) &= n - y(n) \end{aligned}$$

For example:

n	1	2	3	4	5	6	7	8	9	10	11	...
$x(n)$	1	2	2	3	3	3	4	4	4	4	5	...
$y(n)$	0	1	1	3	3	3	6	6	6	6	10	...
$z(n)$	1	1	2	1	2	3	1	2	3	4	1	...
$g(n)$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{1}{2}$	$\frac{3}{1}$	$\frac{2}{2}$	$\frac{1}{3}$	$\frac{4}{1}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{5}{1}$	...

Naturals vs. Rationals (2/2)

- Visualizing $g(n)$.

$$
\begin{array}{llllll}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \cdots & \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \cdots & & \\
\frac{4}{1} & \frac{4}{2} & \cdots & & & \\
\frac{5}{1} & \cdots & & & & \\
\vdots \; \cdots & & & & &
\end{array}
$$

Naturals vs. the Reals

- Let $V = [0, 1)$ be a half-open, interval of real numbers.
- We'll show that $|V| > |\mathbb{N}|$. Clearly $|V| \leq |\mathbb{R}|$ (in fact, $|V| = |\mathbb{R}|$). Thus, this will show that $|\mathbb{R}| > |\mathbb{N}|$.
- The proof is by contradiction.
  - Assume that $|\mathbb{R}| \leq |\mathbb{N}|$.
  - This means that there exists an onto function $g : \mathbb{N} \rightarrow \mathbb{R}$.
  - On the next slide, we'll show that this leads to a contradiction. The argument we use is called a diagonalization argument.
  - g is not onto, a contradiction. This shows that g cannot exist.
  - $\therefore |[0, 1)| > |\mathbb{N}|$. which implies $|\mathbb{R}| > |\mathbb{N}|$.

Diagonalization

- Let $digit(x, k)$ denote the k^{th} digit after the decimal point of x. For example, $digit(0.707106, 4) = 1$, and $digit(\sqrt{\frac{1}{2}}, 40) = 8$.
- Let $y = \sum_{m=1}^{\infty} ((digit(g(m), m) \bmod 8) + 1) \times 10^{-m}$.
  This choice of digits has two handy properties:
  - For all m, $digit(y(m), m) \neq digit(g(m), m)$.
  - All digits are in $\{1, 2, 3, 4, 5, 6, 7, 8\}$. This avoids having to deal with problematic valus for y such as $0.19999999999\ldots$ which has a limit of 0.2, or $0.999999999999\ldots$ which has a limit that is not in $[0, 1)$.
- $y \in [0, 1)$, and $\forall m.\ y \neq g(m)$.
- g is not onto, a contradiction. This shows that g cannot exist.

Diagonalization (2/2)

- Consider the following example of a possible function for g:

m	$g(m)$
0	0.950129285147175
1	0.231138513574288
2	0.606842583541787
3	0.485782468709300
4	0.891288966148902
5	0.762096833027395
6	0.456467465168341
7	0.018503643248224
8	0.821407164295253
9	0.444703364353194
$\vdots$	$\vdots$

- Then y constructed as described on the previous slide will be $0.2378175554\ldots$. Note that for each m, the m^{th} digit of y is different than the m^{th} digit of $g(m)$. Thus, y is guaranteed not to appear on the list.

$|\mathbb{N}|$ vs. $|\mathbb{R}|$

- There is an onto mapping from reals to the naturals, e.g. $\lceil x \rceil$
- Thus, $|\mathbb{R}| \geq |\mathbb{N}|$ by the “pairing” method described above.
- There is no onto mapping from th naturals to the reals. We just showed this.
- Thus, $|\mathbb{N}| \not\geq |\mathbb{R}|$.
- We conclude $|\mathbb{R}| > |\mathbb{N}|$.
- Both are infinite, but there are infinity for the number of reals is inifintely larger than the infinity for the number of naturals (or integers or rationals).

Turing Machines and Languages

- The number of Turing machines is equal to the number of naturals:
  - For any natural number, n, we can define a TM with $n+1$ states. Thus, $|Q|-1$ gives us an onto mapping from TMs to the natural numbers.
  - Any TM is described by a string.
  - We can make an onto mapping from naturals to strings by listing all strings in lexigraphical order.
  - This gives us a onto mapping from integers to TMs.
  - Thus, the number of TMs is the same as the number of naturals.
- The set of languages is the power set of the set of all strings.
  - For any set, S, $|S| < \left|2^S\right|$.
  - Thus, there are more languages than there are TMs.
- $\therefore$ there are languages that are not recognized by any TM.

This coming week (and beyond)

- Reading
  - Today: Sipser, 4.1
  - Oct. 27 (Today): Sipser, 4.2 (midterm cut-off)
  - Oct. 29 (Wednesday): Sipser, 4.2
  - Oct. 31 (Friday): Sipser, 5.1
  - Nov. 3 (A week from Today): Midterm review.
  - Nov. 5 (A week from Wednesday): Midterm 2.
- Homework
  - Oct. 31 (Friday): Homework 7 due, Homework 8 goes out.
    No late homework accepted for homework 7.

    Homework 8 is extra credit.