CpSc 421 Homework 9 **Due: Nov. 17, 4pm**

Attempt any **three** of the **six** problems below. The homework is graded on a scale of 100 points, even though you can attempt fewer or more points than that. Your recorded grade will be the total score on the problems that you attempt.

1. (**30 points**) For each language below, determine whether or not it is Turing decidable. If it is Turing decidable, describe a Turing machine that decides it. If it is not decidable, show this using a reduction from a problem shown to be undecidable in *Sipser*, from lectures. or earlier homework or midterm 2.
   (a) (**10 points)** $\{M \mid M$ describes a TM that halts when run with the empty string for input. $\}$
   (b) (**10 points)** $\{M \mid M$ describes a TM with exactly 42 states. $\}$
   (c) (**10 points)** $\{M \mid M$ describes a TM that accepts exactly 42 strings. $\}$
2. (**30 points**) Same instructions as for problem 1.
   (a) (**10 points)** $\{M \mid M$ describes a TM that decides the halting problem. $\}$
   (b) (**10 points)** $\{M \mid M$ never writes the symbol `0` on two consecutive moves. $\}$
   (c) (**10 points)** $\{M \mid M$ describes a TM that decides every string, w, after at most $\sqrt{|w|} + 12$ moves. $\}$
3. (**35 points**, Sipser problems 5.22, 5.23 and 24)
   (a) (**10 points**) Show that A is Turing-recognizable iff $A \leq_m A_{TM}$.
   (b) (**10 points**) Show that A is Turing-decidable iff $A \leq_m 0^*1^*$.
   (c) (**15 points**) Let $J = \{w \mid$ either $w = 0x$ for some $x \in A_{TM}$ or $w = 1y$ for some $y \notin A_{TM}\}$. Show that neither J nor $\overline{J}$ is Turing-recognizable.
4. (**40 points**, Rice's Theorem: from *Sipser* problem 5.29)

   Rice's Theorem (see *Sipser* problem 5.28): Let

   $$A \quad = \quad \{M \mid M \text{ describes a TM such that } p(M).\ \}$$

   Where p satisifies the following two properties:

   (1) p is non-trivial: there is at least one TM, M_1, such that $p(M_1)$ is true and at least one TM, M_2, such that $p(M_2)$ is false.
   (2) p is a property of the *language* recognized by M.

   Then, A is not Turing decidable.

   Sipser gives a proof for this theorem in the solution to problem 5.28.

   (a) (**2**0 points, *Sipser* problem 5.29) Sipser's proof shows that the two conditions stated above for p are *sufficient* to prove that A is not Turing decidable. Show that both conditions are also *necessary*.
   (b) (**1**0 points) Use Rice's theorem to prove that

   $$B \quad = \quad \{M \mid \text{Every string accepted by } M \text{ has and equal number of } \texttt{a}\text{'s and } \texttt{b}\text{'s.}\ \}$$

   is not Turing decidable.
   (c) (**1**0 points) Use Rice's theorem to prove that

   $$C \quad = \quad \{M \mid M \text{ recognizes a context-free language.}\ \}$$

   is not Turing decidable.

5. (**45 points**) Let

$$E = \{M_1\#M_2 \mid M_1 \text{ and } M_2 \text{ describe TMs such that } L(M_1) = L(M_2).\}$$

Prove that E is complete for class Π_2 of the arithmetic hierarchy (see the Nov. 7 slides). This means that there is a Turing computable reduction from any language in Π_2 to E, and a Turing computable reduction from E to some language in Π_2. You may use the fact that $TOTAL$ is complete for Π_2, where

$$TOTAL = \{M \mid M \text{ is a decider.}\}$$

6. (**50 points**, adapted from *Kozen*) Let $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ be a TM that never overwrites its input. Formally,

$$\begin{aligned}&\mathit{ImmutableInput}(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})\\ &= \forall q, q' \in Q.\ \forall c, c' \in \Gamma.\ \forall d \in \{L, R\}.\ ((\delta(q, c) = (q', c', d)) \wedge (c \in \Sigma)) \Rightarrow (c = c')\end{aligned}$$

M can write anything it wants on the portion of the tape that is initially blank.

(a) (30 points) Prove that for any M with $\mathit{ImmutableInput}(M)$, $L(M)$ is regular.

(b) (10 points) Show that the language

$$F_1 = \{M \mid \mathit{ImmutableInput}(M)\}$$

is Turing decidable.

(c) (10 points) Show that the language

$$F_2 = \{M\#w \mid \mathit{ImmutableInput}(M) \wedge (w \in L(M)\}$$

is not Turing decidable.