

1. (60 points)

- (a) (5 points) Let
- Σ
- be the alphabet
- $\{a, b\}$
- . Give a context free grammar for the language,
- A_1
- , where

$$A_1 = \{w \in \Sigma^* \mid \exists n \in \mathbb{Z}^{\geq 0}. w = a^n b^{2n}\}$$

Note: in all problems, you do not need to write your grammar in CNF or any other “special” form. In fact, you should write your rules so that it is clear why they generate the specified language, and if it is not obvious, add a short explanation of the intuition behind your solution.

- (b) (10 points) Describe a PDA that recognizes language A_1 . You can just draw a transition diagram where edges are labeled as in *Sipser*.
- (c) (15 points) Let Σ be the alphabet $\{a, b\}$. Give a context free grammar for the language, A_2 , where

$$A_2 = \{w \in \Sigma^* \mid \#a(w) = 2\#b(w)\}$$

where $\#a(w)$ denotes the number of a’s in w and likewise for $\#b(w)$. My grammar is fairly short, but it requires a bit of explanation to see that it is correct. Make sure that you include enough of an explanation of why your grammar is correct that your solution is convincing.

- (d) (10 points) Describe a PDA that recognizes language A_2 . You can just draw a transition diagram where edges are labeled as in *Sipser*.
- (e) (10 points) Let Σ be the alphabet $\{a, b, c\}$. Give a context free grammar for the language, A_3 , where

$$A_3 = \{w \in \Sigma^* \mid \exists i, j \in \mathbb{Z}^{\geq 0}. w = a^i b^j c^{i+j}\}$$

- (f) (10 points) Describe a PDA that recognizes language A_3 . You can just draw a transition diagram where edges are labeled as in *Sipser*.

2. (10 points) Prove that language B described below is not context free.

$$B = \{w \in \{a, b\}^* \mid (w = w^{\mathcal{R}}) \wedge (\#a(w) = \#b(w))\}$$

where $w^{\mathcal{R}}$ is the reverse of w . In English, B is the language of all palindromes that contain an equal number of a’s and b’s.

3. (20 points) One of the languages described below is context free and the other is not. Determine which is which. Give a CFG or describe a PDA for the context-free language, and use the pumping lemma to prove that the other language is not context free. For both languages the alphabet is $\{a, b, c, d\}$.

$$C_1 = \{w \mid \exists i, j \in \mathbb{Z}^{\geq 0}. w = a^i b^j c^i d^j\}$$

$$C_2 = \{w \mid \exists i, j \in \mathbb{Z}^{\geq 0}. w = a^i b^j c^j d^i\}$$

4. (20 points) One of the languages described below is context free and the other is not. Determine which is which. Give a CFG or describe a PDA for the context-free language, and use the pumping lemma to prove that the other language is not context free. For both languages the alphabet is $\{a, b, c, d\}$.

$$D_1 = \{x_1 c x_2 c \cdots x_k \mid \text{each } x_i \in \{a, b\}^*, \text{ and for every } i, j \in 1 \dots k, \text{ if } i \neq j, \text{ then } x_i \neq x_j.\}$$

$$D_2 = \{x_1 c x_2 c \cdots x_k \mid \text{each } x_i \in \{a, b\}^*, \text{ there is some pair } i, j \in 1 \dots k \text{ with } i \neq j \text{ and } x_i \neq x_j.\}$$