Extra Credit

Note: All problems on this homework set are extra-credit. You may turn in solutions for up to three of the problems below. Turning in a solution for any part of a problem counts as attempting the entire problem.

BTW, If at the end of the term, someone has a total homework score of $(100 + p)$% (with $p > 0$), I will count it as $100 + p/2$ when computing course grades. This means that you still benefit from the extra credit, but hopefully I'll avoid creating grade inflation in the process.

Have fun!

1. (**25 points**)

   (a) (**5 points**) Draw the state diagram for a DFA or NFA that recognizes the language

   $$A_1 \quad = \quad \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid \#a(w) - \#b(w) \text{ is divisible by three}\},$$

   where $\#a(w)$ denotes the number of a's in w and likewise for $\#b(w)$.

   (b) (**5 points**) Draw the state diagram for a DFA that recognizes the language

   $$A_2 \quad = \quad \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid \text{for every prefix } x \text{ of } w\text{: } |\#a(x) - \#b(x)| \leq 1\}$$

   (c) (**5 points**) Write a regular expression that generates the language A_2 as defined above.

   (d) (**5 points**) Let A_3 be the language

   $$A_3 \quad = \quad \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid \#ab(w) = \#ba(w)\}$$

   where $\#ab(w)$ denotes the number of occurrences of the string ab in w, and likewise for $\#ba(w)$. Draw the state diagram for a NFA that recognizes A_3 or write a regular expression that generates A_3.

   (e) (**5 points**) Let A_4 be the language

   $$A_4 \quad = \quad \{w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^* \mid 2^{|\#a(w)|} - 1 \text{ is divisible by three or } 2^{|\#b(w)|} + 1 \text{ is divisible by five}\}$$

   Draw the state diagram for a DFA that recognizes A_4 or write a regular expression that generates A_4.

2. (**25 points**)

   (a) (**5 points**) Let B_1 be the language of all strings whose length is a power of 2:

   $$B_1 \quad = \quad \{w \mid \exists k \in \mathbb{N}.\ |w| = 2^k\}.$$

   Prove that A is not regular.

   (b) (**20 points**) Consider languages B_2 and B_3 with alphabet $\{\mathtt{a}, \mathtt{b}\}$ defined below:

   $$\begin{array}{rcl} B_2 & = & \{w \mid \#a(w) + \#b(w) = 10\} \\ B_3 & = & \{w \mid \#a(w) - \#b(w) = 10\} \end{array}$$

   One of these langauges is regular, and the other is not. Identify which language is which. For the regular language, show that it is regular by drawing a state diagram for an DFA or NFA that recognizes the language or writing a regular expression that generates it. You don't have to prove your DFA, NFA or regular expression correct, but you do need to write two or three sentences explaining how it works.

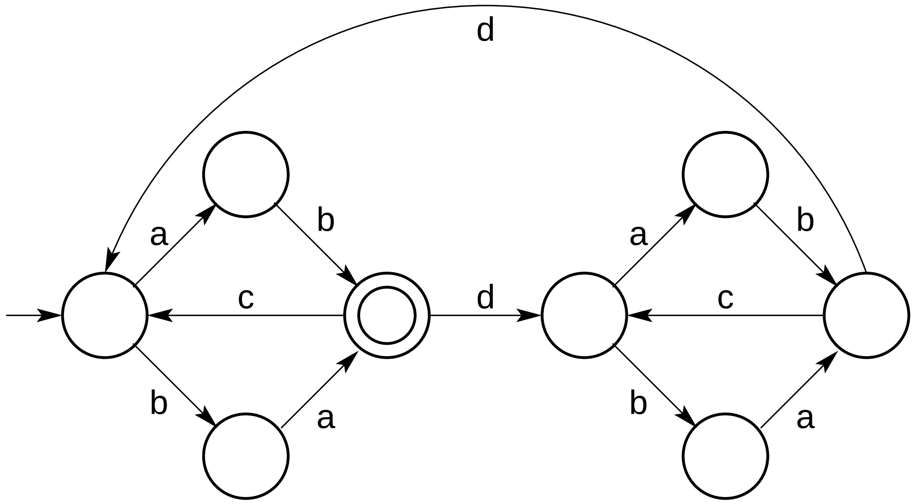

Figure 1: NFA for question 3a

3. (**35 points**)

   (a) (**5 points**) Let C be the language recognized by the NFA depicted in Figure 1. Write a regular expression that generates language C.

   (b) (**15 points**) Prove that for any regular expressions, α, β and γ:

$$(\alpha^*\beta)^*\alpha^* \quad = \quad (\alpha \cup \beta)^*.$$

   (c) (**15 points**) Let α, β and γ be regular expressions. Prove that if $L(\alpha \cup \beta\gamma) \subseteq L(\gamma)$, then $L(\beta^*\alpha) \subseteq L(\gamma)$.

4. (**50 points**) Here is a more general version of the pumping lemma:

   If A is regular, then there is a constant $p > 0$ such that for any string $xyz \in A$ with $|y| \geq p$, then there exist strings u, v, and w such that:

   - $uvw = y$;
   - $|v| > 0$; and
   - $\forall i \geq 0.\ xuv^iwz \in A$

   (a) (**10 points**) Describe a "game with an adversary" (see the September 22 notes) for using this version of the pumping lemma to show that a language is not regular. Make it very clear how the game changes compared with the version in the notes – in particular, how does it change the moves of the person trying to prove that a language is not regular? How does it change the moves of the adversary?

   (b) (**10 points**, Use this more general form of the pumping lemma to prove that the language $\mathsf{a}^*\mathsf{b}^*\mathsf{a}^n\mathsf{b}^n$ is not regular.

   (c) (**15 points**, Write a proof for this more general form of the pumping lemma.

   (d) (**15 points** Show that the language $\mathsf{a}^*\mathsf{b}^*\mathsf{a}^n\mathsf{b}^n$ satisfies the conditions of the simpler version of the pumping lemma presented in the September 22 notes (equivalently, the version from Sipser).

Note: Having stated this form of the pumping lemma, you may use it wherever you find it handy – on homework, tests, etc.

5. (**105 points**) Define

$$\begin{array}{rcl} \mathit{sameLength}(A) & = & \{w \mid \exists x \in A.\ |w| = |x|\} \\ \mathit{square}(A) & = & \{w \mid \exists x \in A.\ |w| = |x|^2\} \\ \mathit{root}(A) & = & \{w \mid \exists x \in A.\ |w| = \sqrt{|x|}\} \\ \mathit{middleThird}(A) & = & \{w \mid \exists x, y, z \in \Sigma^*.\ |x| = |y| = |z|, w = y \text{ and } xyz \in A\} \\ \mathit{outerThirds}(A) & = & \{w \mid \exists x, y, z \in \Sigma^*.\ |x| = |y| = |z|, w = xz \text{ and } xyz \in A\} \\ \mathit{sameAandB}(A) & = & \{w \mid \exists v \in A.\ (\#a(v) = \#a(w)) \wedge (\#b(v) = \#b(w))\} \\ \mathit{sameAorB}(A) & = & \{w \mid \exists v \in A.\ (\#a(v) = \#a(w)) \vee (\#b(v) = \#b(w))\} \end{array}$$

For each statement below, determine whether the statement is true or false and provide a proof.

(a) (**15 points**) If A is regular, then $\mathit{sameLength}(A)$ is regular.

(b) (**15 points**) If A is regular, then $\mathit{square}(A)$ is regular.

(c) (**15 points**) If A is regular, then $\mathit{root}(A)$ is regular.

(d) (**15 points**) If A is regular, then $\mathit{middleThird}(A)$ is regular.

(e) (**15 points**) If A is regular, then $\mathit{outerThirds}(A)$ is regular.

(f) (**15 points**) If A is regular language with alphabet $\{\mathsf{a}, \mathsf{b}\}$, then $\mathit{sameAandB}(A)$ is regular.

(g) (**15 points**) If A is regular language with alphabet $\{\mathsf{a}, \mathsf{b}\}$, then $\mathit{sameAorB}(A)$ is regular.

Hint: four of the seven statements are true.