CpSc 421 Homework 3 **Due: Oct. 3, 4pm**

**NO LATE HOMEWORK ACCEPTED**

1. **(20 points)** Use the pumping lemma to prove that each language listed below is not regular. For each language, I state $\Sigma$ the input alphabet.

   (a) $A_1 = \{w \mid$ the number of zeros in $w$ is less than the number of ones$\}$. $\Sigma = \{0, 1\}$.
   For example, `1`, `011`, and `10100111` are in this language but `0` and `100` are not.

   (b) $A_2 = \mathtt{1}^{2^n}$. $\Sigma = \{\mathtt{1}\}$.
   For example, `1`, `11` and `11111111` are in this language but `111` is not.

   Here's an example of how you can solve a problem like those above:

   Example: $\{w \mid |\#0(w) - \#1(w)| < 4\}$, where $\#0(w)$ is the number of `0`'s in $w$, $\#1(w)$ is the number of `1`'s in $w$, and $\Sigma = \{\mathtt{0}, \mathtt{1}\}$.

   **Solution:** Let $A$ be the language described above and let $p$ be a proposed pumping lemma constant for $A$. Let $w = 0^p 1^{p+3}$. Clearly, $w \in A$. Let $xyz = w$. Let $n = |xy|$. To satisfy the pumping lemma, $n \leq p$; thus $xy = 0^n$. Therefore $xy^0z = 0^{p-|y|}1^{p+3} \notin A$ because $|y| > 1$. Thus, $A$ does not satisfy the pumping lemma and is not regular.
   You can write this with fewer words and say:

   Let $p$ b a propose pumping lemma constant. Let $w = 0^p 1^{p+3}$.
   For any $xyz = w$ with $|xy| \leq p$, $xy^0z = 0^{p-|y|}1^{p+3} \notin A$.
   Therefore, $A$ is not regular.

2. **(20 points)** Let

$$\Sigma_3 \quad = \quad \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \cdots \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

   $\Sigma_3$ contains all size 3 columns of 0s and 1s. A string of symbols in $\Sigma_3$ gives three rows of 0s and 1s. Consider each row to be a binary number with the most significant bit first. For example, let

$$w \quad = \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

   The first row of $w$ is the binary representation of 7, the second row corresponds to 5, and the third row corresponds to 12.

   Let

$$B_+ \quad = \quad \{w \in \Sigma_3^* \mid \text{the bottom row of } w \text{ is the sum of the top two rows}\}.$$

   Show that $B$ is regular.

3. **(20 points)** Let $\Sigma_3$ be defined as in question 2. Let

$$B_\times \quad = \quad \{w \in \Sigma_3^* \mid \text{the bottom row of } w \text{ is the product of the top two rows}\}.$$

   Show that $B$ is not regular.

4. **(20 points)** Consider the two languages described below:

$$C_1 = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid \exists x, y \in \Sigma^*.\ (w = xy) \wedge \#a(x) = \#b(y)\}$$
$$C_2 = \{w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^* \mid \exists x, y \in \Sigma^*.\ (w = x \cdot \mathtt{c} \cdot y) \wedge \#a(x) = \#b(y)\}$$

One of these languages is regular and the other is not. Determine which is which and give short proofs for your conclusions.

5. (**30 points,** from Sipser, problem 2.6)
Give context free grammars generating the following languages:

   (a) **(10 points)** $\{w \mid \exists n \geq 0.\ (w = \mathtt{a}^n\mathtt{b}^{2n}) \vee (\mathtt{w} = \mathtt{a}^{3n}\mathtt{b}^{n})\}$

   (b) **(10 points)** The complement of $\{w \mid \exists n \geq 0.\ w = \mathtt{a}^n\mathtt{b}^n\}$.

   (c) **(10 points)** $\{x_1\mathtt{c}x_2\mathtt{c}\cdots x_k \mid$ each $x_i \in \{\mathtt{a}, \mathtt{b}\}^*$, and for some $i$ and $j$, $x_i = x_j^{\mathcal{R}}\}$.

   For parts (a) and (b), the alphabet is $\{\mathtt{a}, \mathtt{b}\}$. For part (c), the alphabet is $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$.

6. (*20 points, Extra Credit* Consider the language below from the September 22 lecture notes:

$$\begin{aligned} \Sigma &= \{\mathtt{a}, \mathtt{b}, \mathtt{c}\} \\ A &= (\mathtt{aa}^*\mathtt{c})^n(\mathtt{bb}^*\mathtt{c})^n \cup \Sigma^*\mathtt{cc}\Sigma^* \end{aligned}$$

   (a) (*10 points*) Prove that $A$ satisfies the conditions of the pumping lemma as stated in Sipser or the September 22 notes. In other words, show that you can find a constant $p > 0$ such that for any string $w \in A$ with $|w| > p$, you can find strings $x$, $y$ and $z$ such that $w = xyz$ and $xy^iz \in A$ for any $i \geq 0$.

   (b) (*10 points*) Prove that $A$ is not regular.