1. **(20 points)** The September 5 lecture defined a function called f (slide 22):

$$\begin{aligned} f(\epsilon) &= 0 \\ f(s \cdot \mathtt{0}) &= f(s) - 1 \\ f(s \cdot \mathtt{1}) &= f(s) + 1 \end{aligned}$$

   and made four observations about f. Prove the two properties from that slide that are stated below:

   (a) **(10 points)** $f(xy) = f(x) + f(y)$.
   Hint: use induction on the construction (i.e. length) of y.

   (b) If $f(s) > 0$ then for all $k \in [0 \ldots f(s)]$, s can be divided into two strings, x, and y, such that $s = xy$ and $f(x) = k$.
   Hint: my solution uses induction on k and the result from part (a).

2. **(20 points)** Define the set Q inductively as shown below:

$$\begin{aligned} (0,1) &\in Q \\ \text{if } (x,y) \in Q \text{ then } (x+y, y+2) &\in Q \end{aligned}$$

   (a) **(5 points)** Write down the first four tuples in Q (i.e. the four with the smallest values for x or y).

   (b) **(5 points)** Describe the set Q using one English sentence. This should be something along the linese of:

   (x, y) is in Q iff *some property of x and y holds*.

   Your job is to figure out what that property is and state it.

   (c) **(10 points)** Use induction to prove that Q is the set that you described in part (b). Remember to include both directions of the proof.

3. **(25 points)** Let Σ be an alphabet. For $s \in \Sigma^*$ define $s^{\mathcal{R}}$ to be the *reverse* of s as given below:

$$\begin{aligned} \epsilon^{\mathcal{R}} &= \epsilon \\ (w \cdot c)^{\mathcal{R}} &= c \cdot w^{\mathcal{R}} \end{aligned}$$

   a. **(5 points)** Show that $(\mathtt{know})^{\mathcal{R}} = \mathtt{wonk}$ by repeatedly applying the definition of $\mathcal{R}$.

   A string $s \in \Sigma^*$ is a *palindrome* iff $s = s^{\mathcal{R}}$. Let P be the set of all palindromes.

   b. **(5 points)** Give an inductive definition of P.

   c. **(5 points)** Give a short explanation in English of why your definition is correct.
   (This is preparation for the proof requested in part (d).)

   d. **(10 points)** Use induction to prove that the set P that you described in part (b) is indeed the set of all palindromes.