Do problem zero and **six** of problems 1 through 9. If you write down solutions for more that six problems, clearly indicate those that you want graded. Note that problems can be worth 12, 15 or 20 points: you can attempt between 84 and 108 points depending on the problems that you choose. This exam will be graded on a scale of 100 points.

0. (**3 points**) If you have read and understood the instructions in the previous paragraph, write

   I have read the instructions and understand that I am supposed to solve six of the nine problems; that I can thereby attempt between 84 and 108 total points; and that the exam is graded on a scale of 100.

   as your answer to this problem.

1. (**12 points**) Let $\#\mathsf{a}(w)$ be the number of a's in w and $\#\mathsf{b}(w)$ be the number of b's in w. Draw the transition diagram for an NFA that recognizes the language A_1 defined below:

$$A_1 \quad = \quad \{w \in \{\mathsf{a}, \mathsf{b}\}^* \mid \exists x, y, z.\ (w = xyz) \wedge (\#\mathsf{a}(y) \geq \#\mathsf{b}(y) + 3)\}$$

   In other words, the A_1 contains those strings that contain some substring (possibly the entire string itself) that has three more a's than b's.

2. (**20 points**) Let Σ be a finite alphabet. For $x, y \in \Sigma^*$ with $|x| = |y|$, define the *distance* between x and y as the number of symbols for which x and y differ. For those who like formulas:

$$\begin{array}{rcll} \mathit{dist}(\epsilon, \epsilon) & = & 0 & \\ \mathit{dist}(x \cdot c, y \cdot c) & = & \mathit{dist}(x, y), & x, y \in \Sigma^*;\ c \in \Sigma \\ \mathit{dist}(x \cdot c, y \cdot d) & = & \mathit{dist}(x, y) + 1, & x, y \in \Sigma^*;\ c, d \in \Sigma;\ c \neq d \end{array}$$

   Let A be a language. Define

$$\begin{array}{rcl} \mathit{threeStrikes}(A) & = & \{x \mid \exists y \in A\ (|y| = |x|) \wedge (\mathit{dist}(x, y) < 3)\} \\ \mathit{notBad}(A) & = & \{x \mid \exists y \in A.\ (|y| = |x|) \wedge (\mathit{dist}(x, y) < |x|/3)\} \end{array}$$

   (a) (**10 points**) Show that if A is a regular language, then $\mathit{threeStrikes}(A)$ is also regular.
   Hint: My solution has five sentences.

   (b) (**2 points**) Show a language A_1 such that A_1 and $\mathit{notBad}(A_1)$ are both regular.

   (c) (**2 points**) Show another language A_2 such that A_2 is regular and $\mathit{notBad}(A_2)$ is not regular.

   (d) (**6 points**) Prove that for your choice of A_2, $\mathit{notBad}(A_2)$ is not regular.

3. (**12 points**) Let

$$\begin{array}{rcl} B_1 & = & \{x \mid \exists w \in \{\mathsf{a}, \mathsf{b}\}^*.\ x = w\#w^{\mathcal{R}}\} \\ B_2 & = & \{x \mid \exists w \in \{\mathsf{a}, \mathsf{b}\}^*.\ x = w\#w^{\mathcal{R}}\#w\} \end{array}$$

   (a) (**6 points**) Give a CFG for B_1.

   (b) (**6 points**) Prove that B_2 is not a CFL.

4. (**15 points**) As in question 3, let

$$B_2 \quad = \quad \{x \mid \exists w \in \{\mathsf{a}, \mathsf{b}\}^*.\ x = w\#w^{\mathcal{R}}\#w\}$$

   Show that $\overline{B_2}$ is a CFL.

   Hint: Consider the proof that $\overline{a^n b^n c^n}$ is a CFL (e.g, HW 6, Q1.e).

5. (**12 points**) Draw the transition diagram for a Turing machine that erases its tape and then continues from state q_1 with the tape head at the left end of the tape. My solution has three states: q_0 (the initial state); q_1 (the state the TM enters after erasing its tape and moving back to the left end); and q_2 (one more state to get the work done).

6. (**15 points**) Let

$$\begin{array}{rcl} A_1 & = & \{[M] \mid \text{There is some non-empty string } w \text{ such that } [M] \text{ reads every symbol of } w \text{ when run with input } w. \} \\ A_2 & = & \{[M] \mid \text{There is some non-empty string } w \text{ such that } [M] \text{ does not read every symbol of } w \text{ when run with input } w. \} \end{array}$$

   (a) (**3 points**) Is $A_1 = \overline{A_2}$? Give a short justification for your answer.

   (b) (**6 points**) Is A_1 decidable? Justify your answer.

   (c) (**6 points**) Is A_2 decidable? Justify your answer.

7. (**15 points**) Let

$$A_{42} \quad = \quad \{[M] \mid [M] \text{ describes a TM that accepts at least 42 strings. } \}$$

   (a) (**8 points**) Prove that A_{42} is not Turing-decidable.

   (b) (**7 points**) Prove that A_{42} is Turing-recognizable.

8. (**20 points**) A one-counter automaton (OCA) is a 6-tuple $(Q, \Sigma, \delta, q_0, q_f, q_r)$. The symbols $\vdash$ and $\dashv$ are left and right endmarkers; if the input string is w, the OCA's tape will be $\vdash w \dashv$.

   As the name suggests, the OCA has a counter that can hold any integer. The OCA starts in state q_0 with the counter set to zero and the read-head at the leftmost tape square (the one with the $\vdash$). At each step, the OCA makes a move depending on its current state, the tape symbol currently under the read head, and whether or not the value of the counter is equal to zero. Based on this information, the OCA transitions to a new state; moves its tape head one square to the left or the right; and the counter is either incremented, decremented or left unchanged. If the OCA ever reaches state q_a it accepts, and if it reaches q_r it immediately rejects.

   (a) (**5 points**) Describe an OCA that recognizes the language:

$$B_1 \quad = \quad \{x \mid \exists w \in \{\mathsf{a}, \mathsf{b}\}^*.\ x = w\#w^{\mathcal{R}}\}$$

   You don't need to into lots of detail. My solution has nine sentences.

   (b) (**5 points**) Describe an OCA that recognizes the language:

$$B_2 \quad = \quad \{x \mid \exists w \in \{\mathsf{a}, \mathsf{b}\}^*.\ x = w\#w^{\mathcal{R}}\#w\}$$

   You don't need to into lots of detail. My solution two sentences.

   (c) (**10 points**) Prove that the language emptiness problem for OCAs is Turing-undecidable.

   Once again, you don't need lots of detail. Just describe the key points of the reduction so that it's clear that you know how to solve the problem. My solution has seven sentences (using computational histories) or nine (using PCP).

9. (**20 points**) Let A be a regular language with $|A| = \infty$. Prove that there exists a language B with $B \subseteq A$ such that B is not Turing-recognizable.