PDAs and CFLs

Mark Greenstreet, CpSc 421, Term 1, 2006/07

- Every CFL is recognized by some PDA.
- Every PDA recognizes a CFL.

PDAs recognize the CFLs

Proof strategy:

- Every CFL is recognized by a PDA
  - Given a CFG G, construct a PDA P such that $L(P) = L(G)$.
    - Let w be any string in $L(G)$, show that $w \in L(P)$.
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- Every PDA recognizes a CFL
  - Given a PDA P, construct a CFG G such that $L(G) = L(P)$.
    - Let w be any string in $L(P)$, show that $w \in L(G)$.
    - Let w be any string in $L(G)$, show that $w \in L(P)$.

Given a CFG, construct a PDA

- Let G be a CNF CFG.
- If $w \in L(G)$ then w has a leftmost derivation in G.
  - Let $G = (V, \Sigma, R, S_0)$ be a CFG.
  - A leftmost derivation is a sequence of strings, $s_0, s_1, \ldots s_n$ in $(V \cup \Sigma)^*$ such that
    - s_0 = S_0.
    - For each $0 \leq i < n$, we can write $s_i = u_i v_i w_i$ with $u_i \in \Sigma^*$, $v_i \in V$ and $w_i \in (V \cup \Sigma)^*$. In other words, v_i is the leftmost variable in s_i.
    - $(u_i \rightarrow x_i) \in R$ and $s_{i+1} = u_i x_i w_i$.
- We will construct a PDA whose configurations when reading w correspond to a leftmost derivation of w.

Derivations

- Let G be a CFG as before, and $u, v \in (V \cup \Sigma)^*$.
- $uv \overset{*}{\Rightarrow} w$ iff $\exists x, y.(u \overset{*}{\Rightarrow} x) \wedge (v \overset{*}{\Rightarrow} y) \wedge (w = xy)$.
- Proof:
  - $(uv \overset{*}{\Rightarrow} w) \Leftarrow (\exists x, y.(u \overset{*}{\Rightarrow} x) \wedge (v \overset{*}{\Rightarrow} y) \wedge (w = xy))$
  - $(uv \overset{*}{\Rightarrow} w) \rightarrow (\exists x, y.(u \overset{*}{\Rightarrow} x) \wedge (v \overset{*}{\Rightarrow} y) \wedge (w = xy))$

    We prove this by induction on the length of the derivation of w.

Derivations

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- $uv \stackrel{*}{\Rightarrow} w$ iff $\exists x, y. (u \stackrel{*}{\Rightarrow} x) \wedge (v \stackrel{*}{\Rightarrow} y) \wedge (w = xy)$.
- Proof:
  - $(uv \stackrel{*}{\Rightarrow} w) \Leftarrow (\exists x, y. (u \stackrel{*}{\Rightarrow} x) \wedge (v \stackrel{*}{\Rightarrow} y) \wedge (w = xy))$
    - Let x and y be strings in $(V \cup \Sigma)^*$ such that $u \stackrel{*}{\Rightarrow} x$ and $v \stackrel{*}{\Rightarrow} y$.
    - This means there exist strings $\alpha_0 \ldots \alpha_m$ and $\beta_0 \ldots \beta_n$ such that

$$\begin{array}{ll} & (\alpha_0 = u) \wedge (\forall 0 \leq i < m.\ \alpha_i \Rightarrow \alpha_{i+1}) \wedge (\alpha_m = x) \\ \wedge & (\beta_0 = v) \wedge (\forall 0 \leq i < n.\ \beta_i \Rightarrow \beta_{i+1}) \wedge (\beta_n = y) \end{array}$$

    - Let $\begin{array}{lll} \gamma_i & = \alpha_i \beta_0, & 0 \leq i \leq m \\ & = \alpha_m \beta_{i-m}, & m < i \leq m+n \end{array}$
    - By construction,

$$(\gamma_0 = uv) \wedge (\forall 0 \leq i < m.\ \gamma_i \Rightarrow \gamma_{i+1}) \wedge (\gamma_m = xy = w)$$

    - This proves the $\Leftarrow$ direction.
  - $(uv \stackrel{*}{\Rightarrow} w) \rightarrow (\exists x, y. (u \stackrel{*}{\Rightarrow} x) \wedge (v \stackrel{*}{\Rightarrow} y) \wedge (w = xy))$

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Derivations

- Let G be a CFG as before, and $u, v \in (V \cup \Sigma)^*$.
- $uv \stackrel{*}{\Rightarrow} w$ iff $\exists x, y.(u \stackrel{*}{\Rightarrow} x) \wedge (v \stackrel{*}{\Rightarrow} y) \wedge (w = xy)$.
- Proof:
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    We prove this by induction on the length of the derivation of w.
    - Base case: $uv \stackrel{0}{\Rightarrow} w$.
    - Induction step: $uv \stackrel{*}{\Rightarrow} w' \stackrel{1}{\Rightarrow} w$.

Derivations

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  - $(uv \stackrel{*}{\Rightarrow} w) \rightarrow (\exists x, y.(u \stackrel{*}{\Rightarrow} x) \wedge (v \stackrel{*}{\Rightarrow} y) \wedge (w = xy))$
    We prove this by induction on the length of the derivation of w.
    - Base case: $uv \stackrel{0}{\Rightarrow} w$.
      In this case $uv = w$. Let $x = u$ and $y = v$. Clearly $u \stackrel{0}{\Rightarrow} x$ and $v \stackrel{0}{\Rightarrow} y$. This satisfies the claim.
    - Induction step: $uv \stackrel{*}{\Rightarrow} w' \stackrel{1}{\Rightarrow} w$.

Derivations

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    We prove this by induction on the length of the derivation of w.
    - Base case: $uv \stackrel{0}{\Rightarrow} w$.
    - Induction step: $uv \stackrel{*}{\Rightarrow} w' \stackrel{1}{\Rightarrow} w$.
      - By the induction hypothesis, we can find strings x' and y' such that $u \stackrel{*}{\Rightarrow} x'$, $v \stackrel{*}{\Rightarrow} y'$ and $w' = x'y'$.
      - We can write $w' = \alpha g \beta$ such that $g \rightarrow \mu$ and $\alpha \mu \beta = w$.
      - If $|\alpha g| \leq |x'|$ then we can write $x' = \alpha g \gamma$ and note that $w' = \alpha g \gamma y'$.
      - Let $x = \alpha \mu \gamma$ and $y = y'$. Thus, $x' \stackrel{1}{\Rightarrow} x$, $y' \stackrel{0}{\Rightarrow} y$, $w = xy$.
      - We now have x and y such that $u \stackrel{*}{\Rightarrow} x$, $v \stackrel{*}{\Rightarrow} y$, and $w = xy$. This satisifies the claim.

Derivations

- Let G be a CFG as before, and $u, v \in (V \cup \Sigma)^*$.
- $uv \stackrel{*}{\Rightarrow} w$ iff $\exists x, y.(u \stackrel{*}{\Rightarrow} x) \wedge (v \stackrel{*}{\Rightarrow} y) \wedge (w = xy)$.
- Proof:
  - $(uv \stackrel{*}{\Rightarrow} w) \Leftarrow (\exists x, y.(u \stackrel{*}{\Rightarrow} x) \wedge (v \stackrel{*}{\Rightarrow} y) \wedge (w = xy))$
  - $(uv \stackrel{*}{\Rightarrow} w) \rightarrow (\exists x, y.(u \stackrel{*}{\Rightarrow} x) \wedge (v \stackrel{*}{\Rightarrow} y) \wedge (w = xy))$
    We prove this by induction on the length of the derivation of w.
    - Base case: $uv \stackrel{0}{\Rightarrow} w$.
    - Induction step: $uv \stackrel{*}{\Rightarrow} w' \stackrel{1}{\Rightarrow} w$.
    - This proves the $\Rightarrow$ direction.

Leftmost Derivations (1/2)

- Let $u \in (V \cup \Sigma)^*$ and $u \overset{*}{\Rightarrow} w$ where $w \in \Sigma^*$.
- Then, $u \overset{*}{\Rightarrow} w$ by a leftmost derivation.
- Proof:
  - Let $s_0, s_1, \ldots s_n$ be a derivation of w.
  - If this is a leftmost derivation, then we're done.
  - Otherwise, choose i such that s_0 through s_i is a leftmost derivation, and $s_i \Rightarrow s_{i+1}$ is not leftmost.
  - We'll show how we can make an equivalent derivation where the first $i + 1$ steps are leftmost and the total number of steps are unchanged.

Leftmost Derivations (2/2)

- Choose i such that s_0 through s_i is a leftmost derivation, and $s_i \Rightarrow s_{i+1}$ is not leftmost.
  - Let $s_i = ugv$ where $u \in \Sigma^*$, $g \in V$, and $v \in (V \cup \Sigma)^*$. In other words, g is the first variable in s_i.
  - As shown above, we can find x and y such that $ug \stackrel{*}{\Rightarrow} x$, $v \stackrel{*}{\Rightarrow} y$ and $w = xy$.
  - This means there exist strings $\alpha_0 \ldots \alpha_m$ and $\beta_0 \ldots \beta_n$ such that

$$\begin{array}{ll} & (\alpha_0 = ug) \wedge (\forall 0 \leq j < m.\ \alpha_j \Rightarrow \alpha_{j+1}) \wedge (\alpha_m = x) \\ \wedge & (\beta_0 = v) \wedge (\forall 0 \leq k < n.\ \beta_i \Rightarrow \beta_{k+1}) \wedge (\beta_n = y) \end{array}$$

  - Let $\begin{array}{rcll} s'_j & = & s_j, & 0 \leq j \leq i \\ & = & \alpha_{j-i} v, & i < j \leq i+m \\ & = & x\beta_{j-(i+m)}, & i+m < j \leq i+m+n \end{array}$
  - The sequence $s'_0, \ldots, s'_{i+m+n}$ is a derivation of w that is leftmost for (at least) its first $i+1$ steps.
- We can continue this process until we get a leftmost derivation.

CNF and Leftmost Derivations

- If $G = (V, \Sigma, R, S_0)$ is a CNF grammar, and $S_0 \stackrel{*}{\Rightarrow}_{left} \alpha$, then $\alpha \in \Sigma^* \circ V^*$, where $\stackrel{*}{\Rightarrow}_{left}$ denotes a leftmost derivation.
- Proof, by induction on the length of the derivation.
  - Base case: $S_0 \stackrel{0}{\Rightarrow}_{left} \alpha$.
    - $\alpha = S_0$.
    - Let $s = \epsilon$ and $v = S_0$.
    - The claim is satisfied.
  - Induction step: $S_0 \stackrel{*}{\Rightarrow}_{left} \alpha' \stackrel{1}{\Rightarrow}_{left} \alpha$.
    - By the induction hypothesis, $\alpha' = s'v'$ with $s' \in \Sigma^*$ and $v' \in V^*$.
    - Let $v' = gv"$ with $g \in V$ and $v" \in V^*$.
    - There is some rule, $g \rightarrow \beta$ in R such that $w = s'\beta v"$.
    - Note that $s' \in \Sigma^*$ and $v" \in V^*$. Furthermore, either $\beta \in \Sigma$, $\beta \in V^2$, or $g = S_0$ and $\beta = \epsilon$. In each of these cases, $s'\beta v" \in \Sigma^* \circ V^*$.

Given a CFG, construct a PDA

- Let $G = (V, \Sigma, R, S_0)$ be a CNF CFG.
- The main idea:
  - Construct a PDA, P, whose operation when reading w corresponds to a leftmost derivation of w.
  - Each step of the leftmost derivation produces a string of the form sv where $s \in \Sigma^*$ and $v \in V$.
  - s corresponds to the input read so far.
  - The PDA represents the string v with the sequence of symbols on its stack.
- More formally, ...

Given a CFG, construct a PDA

- Let $G = (V, \Sigma, R, S_0)$ be a CNF CFG.
- The main idea: ...
- More formally,

$$
\begin{array}{rcll}
P & = & (Q, \Sigma, V \cup \{\$\}, \delta, q_0, \{q_3\}), & \text{the PDA} \\
Q & = & \{q_0, q_1, q_2, q_3\} \cup \{q_v \mid \text{for each } v \in V\}, & \text{the states} \\
\delta(q_0, \epsilon, \epsilon) & = & \{(q_1, \$)\}, \quad \text{first move: push } \$ & \\
\delta(q_1, \epsilon, \epsilon) & = & \{(q_2, S_0)\}, \quad \text{second move: push } S_0 & \\
\delta(q_2, c, A) & = & \{(q_2, \epsilon)\}, \quad \text{if } (A \rightarrow c) \in R & \\
\delta(q_2, \epsilon, A) & = & \{(q_X, Y)\}, \quad \text{if } (A \rightarrow XY) \in R & \\
\delta(q_X, \epsilon, \epsilon) & = & \{(q_2, X)\}, \quad \text{push } X & \\
\delta(q_2, \epsilon, \$) & = & \{(q_3, X)\}, \quad \text{accept} &
\end{array}
$$

- Claim: $L(P) = L(G)$.

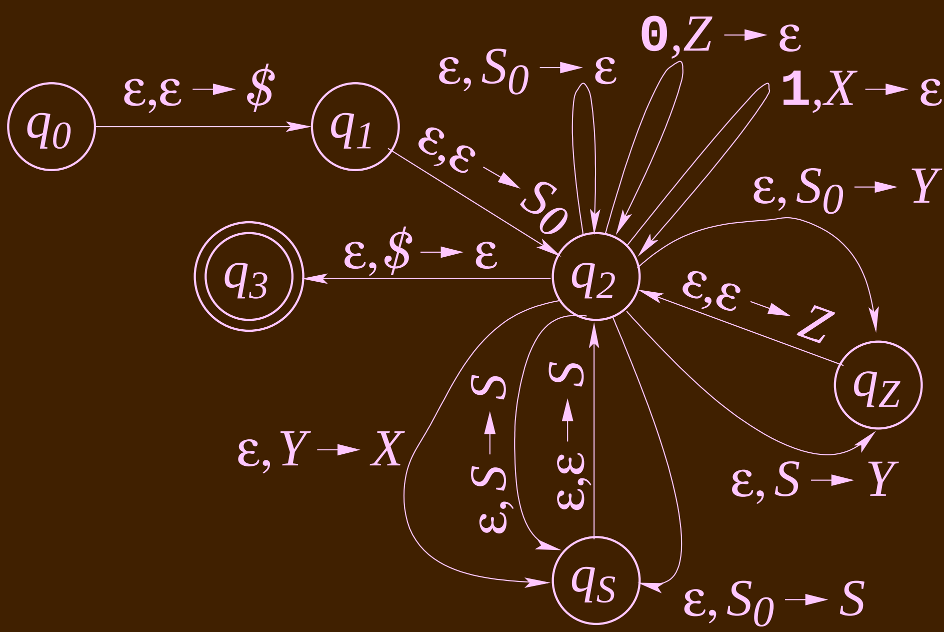
An Example

- The Balanced Parentheses Language:

$$S \rightarrow \epsilon \mid \mathtt{0}S\mathtt{1} \mid SS$$

- In CNF

$$\begin{aligned} S_0 &\rightarrow \epsilon \mid ZY \mid SS \\ S &\rightarrow ZY \mid SS \\ Z &\rightarrow \mathtt{0} \\ Y &\rightarrow SX \\ X &\rightarrow \mathtt{1} \end{aligned}$$



$L(G) \subseteq L(P)$

- Let $w \in L(G)$.
- Let $x_2, x_3, \ldots x_n$ be a leftmost derivation of w.
- For each x_i, let $x_i = s_i v_i$ with $s_i \in \Sigma^*$ and $v_i \in V^*$. Let $y_i \in \Sigma^*$ such that $x_i y_i = w$; in other words, y_i is the unread input.
- Let

$$
\begin{aligned}
f(i) &= i, & & \text{if } i \leq 2 \\
&= f(i-1)+1, & & \text{if } i > 2 \text{ and } x_{i-1} \Rightarrow x_i \text{ by a rule of the form} \\
& & & \qquad X \rightarrow c \\
&= f(i-1)+2, & & \text{if } i > 2 \text{ and } x_{i-1} \Rightarrow x_i \text{ by a rule of the form} \\
& & & \qquad X \rightarrow YZ
\end{aligned}
$$

- Let $C_0, C_1, C_3 \ldots C_{f(n)+1}$ be a sequence of configurations of P . . .

$L(G) \subseteq L(P)$

- Let $w \in L(G)$.
- Let x_2, x_3, ... x_n be a leftmost derivation of w.
- For each x_i, let $x_i = s_i v_i$ with $s_i \in \Sigma^*$ and $v_i \in V^*$. Let $y_i \in \Sigma^*$ such that $x_i y_i = w$; in other words, y_i is the unread input.
- Let $C_0, C_1, C_3 \ldots C_{f(n)+1}$ be a sequence of configurations of P :

C_0	$=$	(q_0, w, ϵ),	the initial configuration
C_1	$=$	$(q_1, w, \$)$,	end-marker on stack
C_2	$=$	$(q_2, w, S_0\$)$,	start symbol on stack
$C_{f(i)}$	$=$	$(q_2, y_i, v_i \cdot \$)$,	if $2 \leq i \leq n$
$C_{f(i)+1}$	$=$	$(q_Y, y_i, Z \cdot v_i \cdot \$)$,	if $2 \leq i \leq n$ and $x_i \overset{1}{\Rightarrow} x_{i+1}$ by $X \rightarrow YZ$
$C_{f(n)+1}$	$=$	$(q_3, \epsilon, \epsilon)$,	pop $\$$ and accept

$L(G) \subseteq L(P)$

- Let $w \in L(G)$.
- Let x_2, x_3, $\ldots x_n$ be a leftmost derivation of w.
- For each x_i, let $x_i = s_i v_i$ with $s_i \in \Sigma^*$ and $v_i \in V^*$. Let $y_i \in \Sigma^*$ such that $x_i y_i = w$; in other words, y_i is the unread input.
- Let $C_0, C_1, C_3 \ldots C_{f(n)+1}$ be a sequence of configurations of $P\ldots$
- $C_0, C_1, C_3 \ldots C_{f(n)+1}$ is a legal sequence of configurations for P, and $C_{f(n)+1}$ is an accepting configuration.
- $\therefore w \in L(P)$.

The Example (again, 1/6)

- CNF grammar for balanced parantheses, G:

$$
\begin{aligned}
S_0 &\rightarrow \epsilon \mid ZY \mid ZX \mid SS \\
S &\rightarrow ZY \mid ZX \mid SS \\
Z &\rightarrow \texttt{0} \\
Y &\rightarrow SX \\
X &\rightarrow \texttt{1}
\end{aligned}
$$

- $\texttt{00100111} \in L(G)$

derivation		configurations				
	$\rightarrow$	(	$q_0,$	$\texttt{00100111},$	ϵ	)
	$\rightarrow$	(	$q_1,$	$\texttt{00100111},$	$\$$	)
S_0	$\rightarrow$	(	$q_2,$	$\texttt{00100111},$	$S_0\$$	)
$\Rightarrow ZY$	$\rightarrow$	(	$q_Z,$	$\texttt{00100111},$	$Y\$$	)
	$\rightarrow$	(	$q_2,$	$\texttt{00100111},$	$ZY\$$	)
$\Rightarrow \texttt{0}Y$	$\rightarrow$	(	$q_2,$	$\texttt{0100111},$	$Y\$$	)

The Example (again, 2/6)

- CNF grammar for balanced parantheses, G:

$$
\begin{aligned}
S_0 &\rightarrow \epsilon \mid ZY \mid ZX \mid SS \\
S &\rightarrow ZY \mid ZX \mid SS \\
Z &\rightarrow \texttt{0} \\
Y &\rightarrow SX \\
X &\rightarrow \texttt{1}
\end{aligned}
$$

- $\texttt{00100111} \in L(G)$

derivation	configurations
$\Rightarrow \texttt{0}Y$	$\rightarrow (q_2, \texttt{0100111}, Y\$)$
$\Rightarrow \texttt{0}SX$	$\rightarrow (q_S, \texttt{0100111}, X\$)$
	$\rightarrow (q_2, \texttt{0100111}, SX\$)$
$\Rightarrow \texttt{0}SSX$	$\rightarrow (q_S, \texttt{0100111}, SX\$)$
	$\rightarrow (q_2, \texttt{0100111}, SSX\$)$
$\Rightarrow \texttt{0}ZXSX$	$\rightarrow (q_Z, \texttt{0100111}, XSX\$)$

The Example (again, 3/6)

- CNF grammar for balanced parantheses, G:

$$
\begin{aligned}
S_0 &\rightarrow \epsilon \mid ZY \mid ZX \mid SS \\
S &\rightarrow ZY \mid ZX \mid SS \\
Z &\rightarrow \texttt{0} \\
Y &\rightarrow SX \\
X &\rightarrow \texttt{1}
\end{aligned}
$$

- $\texttt{00100111} \in L(G)$

derivation	configurations
$\Rightarrow\ \texttt{0}ZXSX$	$\rightarrow (q_Z, \texttt{0100111}, XSX\$)$
	$\rightarrow (q_2, \texttt{0100111}, ZXSX\$)$
$\Rightarrow\ \texttt{00}XSX$	$\rightarrow (q_2, \texttt{100111}, XSX\$)$
$\Rightarrow\ \texttt{001}SX$	$\rightarrow (q_2, \texttt{00111}, SX\$)$
$\Rightarrow\ \texttt{001}ZYX$	$\rightarrow (q_Z, \texttt{00111}, YX\$)$

The Example (again, 4/6)

- CNF grammar for balanced parantheses, G:

$$
\begin{aligned}
S_0 &\rightarrow \epsilon \mid ZY \mid ZX \mid SS \\
S &\rightarrow ZY \mid ZX \mid SS \\
Z &\rightarrow \texttt{0} \\
Y &\rightarrow SX \\
X &\rightarrow \texttt{1}
\end{aligned}
$$

- $\texttt{00100111} \in L(G)$

derivation	configurations
$\Rightarrow \texttt{001}ZYX$	$\rightarrow (q_Z, \texttt{00111}, YX\$)$
	$\rightarrow (q_2, \texttt{00111}, ZYX\$)$
$\Rightarrow \texttt{0010}YX$	$\rightarrow (q_2, \texttt{0111}, YX\$)$
$\Rightarrow \texttt{0010}SXX$	$\rightarrow (q_S, \texttt{0111}, XX\$)$
	$\rightarrow (q_2, \texttt{0111}, SXX\$)$
$\Rightarrow \texttt{0010}ZXXX$	$\rightarrow (q_Z, \texttt{0111}, XXX\$)$

The Example (again, 5/6)

- CNF grammar for balanced parantheses, G:

$$
\begin{aligned}
S_0 &\rightarrow \epsilon \mid ZY \mid ZX \mid SS \\
S &\rightarrow ZY \mid ZX \mid SS \\
Z &\rightarrow \texttt{0} \\
Y &\rightarrow SX \\
X &\rightarrow \texttt{1}
\end{aligned}
$$

- $\texttt{00100111} \in L(G)$

	derivation		configurations
$\Rightarrow$	$\texttt{0010}ZXXX$	$\rightarrow$	$(\ q_Z,\ \texttt{0111},\ XXX\$\ )$
		$\rightarrow$	$(\ q_2,\ \texttt{0111},\ ZXXX\$\ )$
$\Rightarrow$	$\texttt{00100}XXX$	$\rightarrow$	$(\ q_2,\ \texttt{111},\ XXX\$\ )$
$\Rightarrow$	$\texttt{001001}XX$	$\rightarrow$	$(\ q_2,\ \texttt{11},\ XX\$\ )$
$\Rightarrow$	$\texttt{0010011}X$	$\rightarrow$	$(\ q_2,\ \texttt{1},\ X\$\ )$
$\Rightarrow$	$\texttt{00100111}$	$\rightarrow$	$(\ q_2,\ \texttt{1},\ \$\ )$

The Example (again, 6/6)

- CNF grammar for balanced parantheses, G:

$$\begin{array}{rcl} S_0 & \rightarrow & \epsilon \mid ZY \mid ZX \mid SS \\ S & \rightarrow & ZY \mid ZX \mid SS \\ Z & \rightarrow & \texttt{0} \\ Y & \rightarrow & SX \\ X & \rightarrow & \texttt{1} \end{array}$$

- $\texttt{00100111} \in L(G)$

$$\begin{array}{cc|cccccc} & \text{derivation} & & & \text{configurations} & & & \\ \Rightarrow & \texttt{00100111} & \rightarrow & (& q_2, & \texttt{1}, & \$ &) \\ & & \rightarrow & (& q_3, & \texttt{1}, & &) \end{array}$$