CpSc 421 Homework 3 **Due: Oct. 6, 4pm**

NO LATE HOMEWORK ACCEPTED

1. **(40 points)** Use the pumping lemma to prove that each language listed below is not regular. For each language, I state Σ the input alphabet.

   (a) $\{w \mid$ the number of zeros in w is less than the number of ones$\}$. $\Sigma = \{0, 1\}$.
   For example, `1`, `011`, and `10100111` are in this language but `0` and `100` are not.

   (b) $\mathtt{1}^{n^2}$. $\Sigma = \{\mathtt{1}\}$.
   For example, ϵ `1` and `1111111111111111` are in this language but `111` is not. in this language but `0` and `100` are not.

   (c) $\{w \cdot \mathtt{c} \cdot w^{\mathcal{R}} \mid w \in \{\mathtt{a}, \mathtt{b}\}^*, w^R$ is the reverse of $w\}$. $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$.
   For example, `abcba`, `c` and `aababbcbbabaa` are in this language, but `abcab`, ϵ, `abbbba` and `abbcbaa` are not.

   (d) $\{w \mid$ the number of left parentheses in any prefix of w is greater than or equal to the number of right parentheses, and the number of left parentheses in w is equal to the number of right parentheses $\}$.
   $\Sigma = \{\mathtt{(}, \mathtt{)}\}$.
   For example, `()`, ϵ and `((()(()()))())` are in this language, but `(()`, `)(`, and `(()))` are not.

   Here's an example of how you can solve a problem like those above:

   Example: $\{w \mid |\#0(w) - \#1(w)| < 4\}$, where $\#0(w)$ is the number of `0`'s in w, $\#1(w)$ is the number of `1`'s in w, and $\Sigma = \{\mathtt{0}, \mathtt{1}\}$.

   Solution: Let A be the language described above and let p be a proposed pumping lemma constant for A. Let $w = 0^p 1^{p+3}$. Clearly, $w \in A$. Let $xyz = w$. Let $n = |xy|$. To satisfy the pumping lemma, $n \leq p$; thus $xy = 0^n$. Therefore $xy^0z = 0^{p-|y|}1^{p+3} \notin A$ because $|y| > 1$. Thus, A does not satisfy the pumping lemma and is not regular.
   You can write this with fewer words and say:

   Let p b a propose pumping lemma constant. Let $w = 0^p 1^{p+3}$.
   For any $xyz = w$ with $|xy| \leq p$, $xy^0z = 0^{p-|y|}1^{p+3} \notin A$.
   Therefore, A is not regular.

2. **(20 points)** Let

$$\Sigma_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \cdots \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

   Σ_3 contains all size 3 columns of 0s and 1s. A string of symbols in Σ_3 gives three rows of 0s and 1s. Consider each row to be a binary number with the most significant bit first. For example, let

$$w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

   The first row of w is the binary representation of 7, the second row corresponds to 5, and the third row corresponds to 12.

   Let

$$B = \{w \in \Sigma_3^* \mid \text{the bottom row of } w \text{ is the product of the top two rows}\}.$$

   Show that B is not regular.

3. **(20 points)** Consider the two languages described below:

- $\{w \in \{\mathsf{a}, \mathsf{b}\}^* \mid \exists x, y \in \Sigma^*.\ (w = xy) \wedge \#a(x) = \#b(y)\}$

- $\{w \in \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}^* \mid \exists x, y \in \Sigma^*.\ (w = x \cdot \mathsf{c} \cdot y) \wedge \#a(x) = \#b(y)\}$

One of these languages is regular and the other is not. Determine which is which and give short proofs for your conclusions.

4. **(30 points)** (Sipser problem 1.47)
If A is any language over alphabet Σ, let $A_{\frac{1}{2}-}$ be the set of all first halves of strings in A so that

$$A_{\frac{1}{2}-} \quad = \quad \{u \mid \exists v \in \Sigma^{|u|}.\ uv \in A\}$$

Show that if A is regular, then so is $A_{\frac{1}{2}-}$.