1. **(25 points)**: (from Sipser, problem 1.6) Give state diagrams of DFAs recognizing the following languages. In all parts the alphabet is $\{0,1\}$.

   (a) $\{w \mid w$ begins with a 1 and ends with a 0$\}$.

   (b) $\{w \mid w$ contains at least three 1s$\}$.

   (c) $\{w \mid w$ contains the substring 0101, i.e., $w = x0101y$ for some x and $y\}$.

   (d) $\{w \mid w$ has length at least 3 and its third symbol is a 0$\}$.

   (e) $\{w \mid w$ starts with 0 and has odd length, or starts with 1 and has even length$\}$.

2. **(30 points)**: (from Sipser, problem 1.7) Give state diagrams of NFAs with the specified number of states recognizing each of the following languages. In all parts the alphabet is $\{0,1\}$.

   (a) The language of $\{w \mid w$ contains the substring 0101, i.e., $w = x0101y$ for some x and $y\}$ with five states.

   (b) The language of $\{w \mid w$ contains an even number of 0s, or contains exactly two 1s$\}$ with six states.

   (c) The language $\{0\}$ with two states.

   (d) The language $0^*1^*0^+$ with three states.

   (e) The language $\{\epsilon\}$ with one state.

   (f) The language 0^* with one state.

   Note: 0^* is a string of *zero* or more 0s.
   0^+ is a string of *one* or more 0s.

3. **(25 points)**: Closure properties of regular languages.

   (a) **(10 points)** Prove that the regular languages are closed under complement. In other words, show that if L is a regular language, then the language $\overline{L}$ is regular as well.

   (b) **(15 points)** Prove that the regular languages are closed under intersection. In other words, show that if L_1 and L_2 are regular languages, then $L_1 \cap L_2$ is regular as well.

4. **(25 points)**: Let $M = \{Q, \{\mathtt{a}\}, \delta, q_0, F)$ be a finite automaton. Note that the alphabet for M has only one symbol, $\mathtt{a}$. All strings in $\{\mathtt{a}\}^*$ have the form $\mathtt{a}^m$ for some $m \in \mathbb{N}$.

   Prove that there are sets $A, B \subset \mathbb{N}$, and an integer, k, such that:

   - $(a^m \in L(M)) \quad \Leftrightarrow \quad (m \in A) \vee (\exists i \in B.\ \exists j \in \mathbb{N}.\ m = i + j * k)$.
     In English, this says that the length of any string in $L(M)$ is either given by an element of A or is the sum of an element of B and a multiple of k.
   - Every element of A or B is at most $|Q|$.
   - Likewise, $k \leq |Q|$.

   Hint: Think of what the state-transition diagram for M must look like, and consider what M does while reading a string, a^m.