CpSc 421 Homework 4 **Due: Nov. 7**

NO LATE HOMEWORK ACCEPTED

1. **(30 points):** Binary multiplication.
   As in homework 2, let $\Sigma = \{0,1\}^3$, i.e. the set of tuples consisting of three bits. For, $(a,b,c) \in \Sigma$, define

$$\begin{aligned} \mathsf{first}((a,b,c)) &= a \\ \mathsf{second}((a,b,c)) &= b \\ \mathsf{third}((a,b,c)) &= c \end{aligned}$$

   We overload first, second, and third to strings as shown below:

$$\begin{aligned} \mathsf{first}(\epsilon) &= \epsilon \\ \mathsf{first}(x \cdot \mathsf{c}) &= \mathsf{first}(x) \cdot \mathsf{first}(\mathsf{c}) \\[4pt] \mathsf{second}(\epsilon) &= \epsilon \\ \mathsf{second}(x \cdot \mathsf{c}) &= \mathsf{second}(x) \cdot \mathsf{second}(\mathsf{c}) \\[4pt] \mathsf{third}(\epsilon) &= \epsilon \\ \mathsf{third}(x \cdot \mathsf{c}) &= \mathsf{third}(x) \cdot \mathsf{third}(\mathsf{c}) \end{aligned}$$

   For example, if $s = (0,0,0)(0,0,1)(0,1,0)(0,1,1)(1,0,0)(1,0,1)$, then $\mathsf{first}(s) = 000011$, $\mathsf{second}(s) = 001100$, and $\mathsf{third}(s) = 010101$. For $s \in \{0,1\}^*$, let $\mathsf{binary}(s)$ denote the binary value of s when the most significant bit is the first symbol of the string:

$$\begin{aligned} \mathsf{binary}(\epsilon) &= 0 \\ \mathsf{binary}(0) &= 0 \\ \mathsf{binary}(1) &= 1 \\ \mathsf{binary}(x \cdot \mathsf{c}) &= 2 * \mathsf{binary}(x) + \mathsf{binary}(c) \end{aligned}$$

   Let B denote the language of binary multiplication:

$$B = \{w \mid \mathsf{binary}(\mathsf{third}(w)) = \mathsf{binary}(\mathsf{first}(w)) * \mathsf{binary}(\mathsf{second}(w))\}$$

   (a) **(10 points):** Use the pumping lemma for regular languages to show that B is not regular.

   (b) **(20 points):** Is B context-free? Prove your answer.

2. **(30 points):** Let $\Sigma = \{a,b\}$. As usual, let $\#\mathsf{a}(x)$ denote the number of occurences of a in x, and let $\#\mathsf{b}(x)$ denote the number of occurences of b. Show that each of the languages below is context-free:

   (a) **(10 points)**: $\{x \mid \#\mathsf{a}(x) \leq \#\mathsf{b}(x)\}$

   (b) **(10 points)**: $\left\{x \,\middle|\, |\#\mathsf{a}(x) - \#\mathsf{b}(x)| < 3\right\}$

   (c) **(10 points)**: $\left\{x \,\middle|\, (|\#\mathsf{a}(x) - \#\mathsf{b}(x)| \bmod 3) = 2\right\}$

3. **(20 points):** (Kozen, Miscellaneous Exercise 31)
   Let $\Sigma = \{a, b, \$\}$. One of the two sets is regular and the other is not. Which is which? Prove your answers.

   (a) **(10 points)**: $\{w \mid \exists x, y \in \{a,b\}^*.\ (w = xy) \wedge (\#\mathsf{a}(x) = \#\mathsf{b}(y))\}$

   (b) **(10 points)**: $\{w \mid \exists x, y \in \{a,b\}^*.\ (w = x\$y) \wedge (\#\mathsf{a}(x) = \#\mathsf{b}(y))\}$

4. **(20 points):** (Kozen, Miscellaneous Exercise 75)
   Define a context-free grammar for regular expressions over an alphabet Σ. Your grammar should have the terminal symbols $\Sigma \cup \{\epsilon, \phi, \cdot, +, (,), {}^*\}$. Your grammar should be unambiguous.