Sorting Networks

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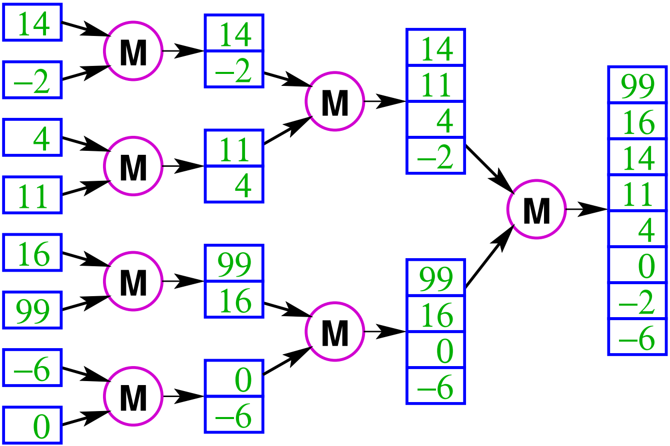
CpSc 418 – Nov. 5, 2013

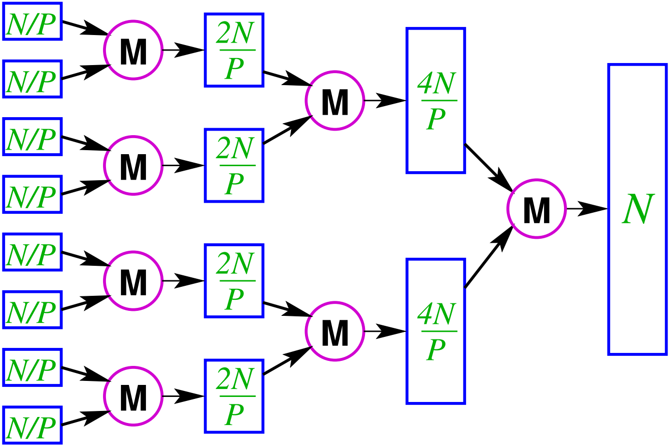
Lecture Outline

- Parallelizing mergesort and/or quicksort
- Sorting Networks
- Bitonic Sorting

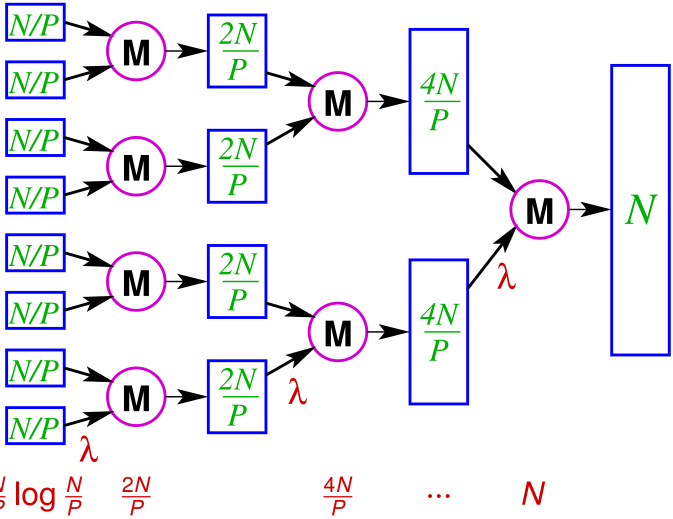
Parallelizing Mergesort



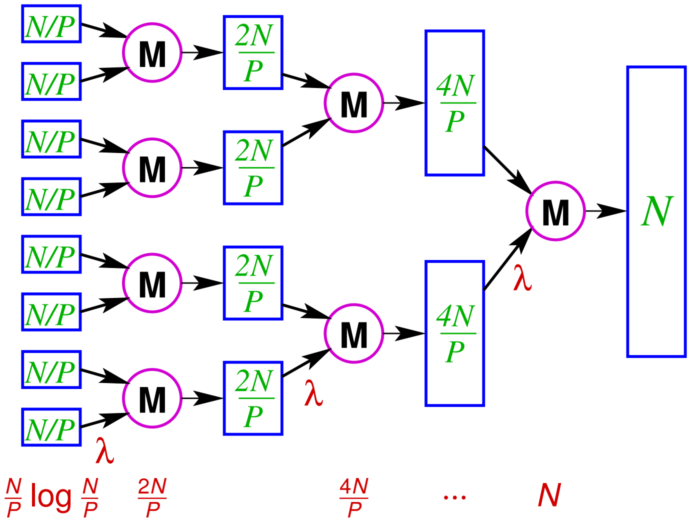
Parallelizing Mergesort



Parallelizing Mergesort



Parallelizing Mergesort

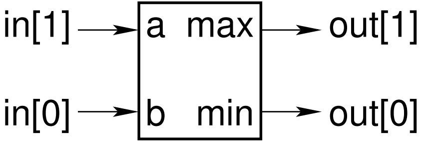


Total time: $\frac{N}{P}(\log N + 2(P-1) - \log P) + (\log P)\lambda$

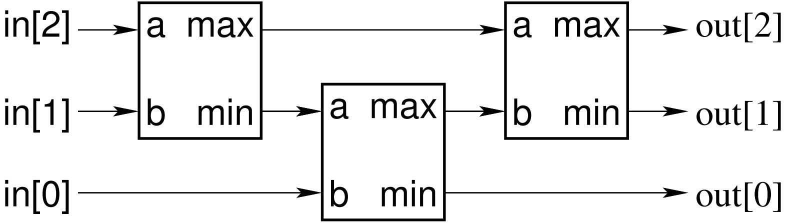
Parallelizing Quicksort

Sorting Networks

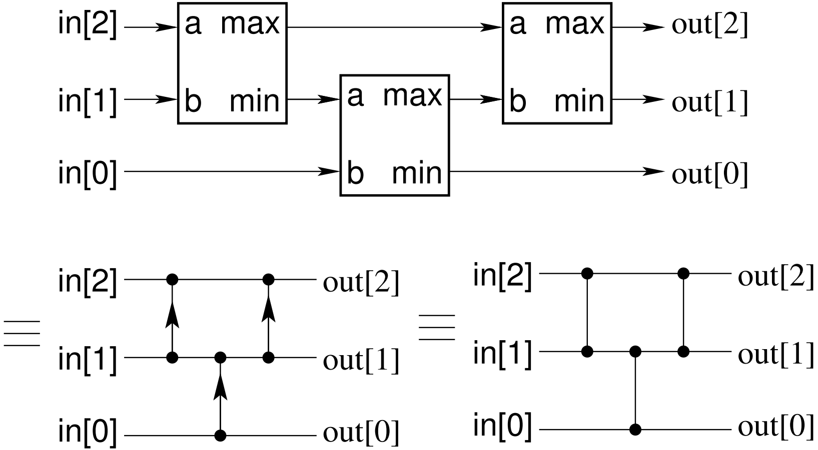
Sorting Network for 2–elements



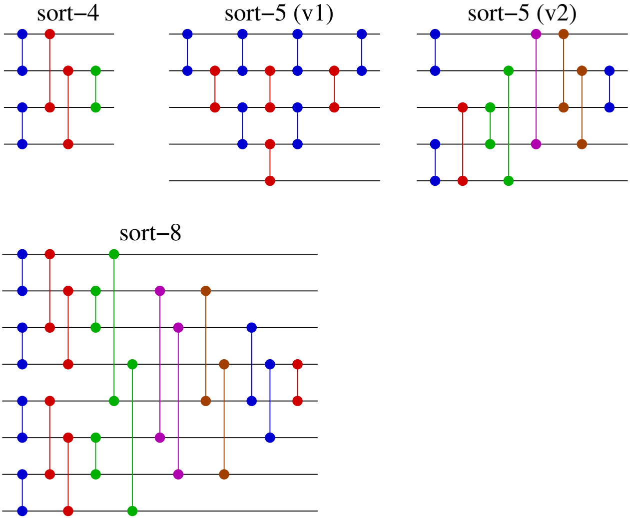
A Sorting Network for 3–elements



Sorting Networks – Drawing



Sorting Networks – Examples



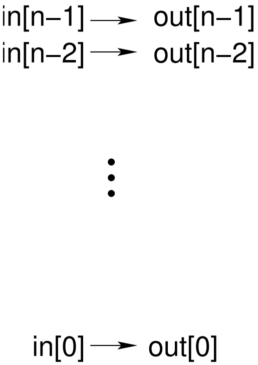
See: http://pages.ripco.net/~jgamble/nw.html

Sorting Networks: Definition

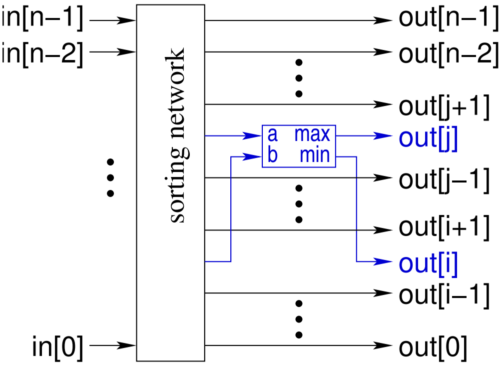
Structural version:

An N-input sorting network is either:

The identity function

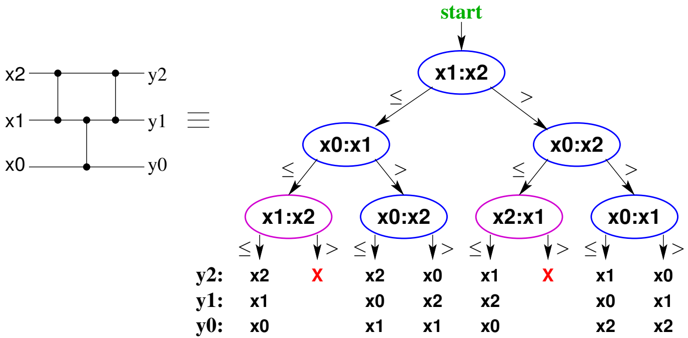


A sorting network composed with a compare-and-swap element



Sorting Networks: Definition

Decision-tree version:



- Let v be an arbitrary vertex of a decision tree, and let x_i and x_j be the variables compared at vertex v.
- A decision tree is a sorting network iff for every such vertex, the left subtree is the same as the right subtree with x_i and x_j exchanged.

The 0-1 Principle

If a sorting network correctly sorts all inputs consisting only of 0's and 1's, then it correctly sorts inputs consisting of arbitrary (comparable) values.

Monotonicity Lemma

Lemma: sorting networks commute with monotonic functions.

- Let $\mathbb{D}$ and $\mathbb{E}$ be two domains, each with an ordering relation.
- $f : \mathbb{D} \rightarrow \mathbb{E}$ is monotonic iff

$$\forall x, y \in \mathbb{D}.\ x \leq y \rightarrow f(x) \leq f(y)$$

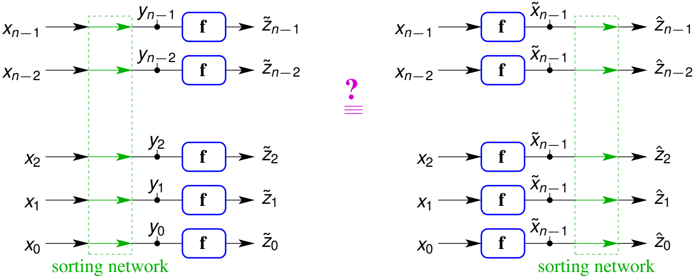
- We extend f element-wise to vectors:

$$f([x_0, x_1, \ldots, x_{n-1}]) \quad = \quad [f(x_0), f(x_1), \ldots, f(x_{n-1})]$$

- We can view an n-input sorting network, S as a function on vectors of length n.
- The monotonicity lemma states that $f \bullet S \;\equiv\; S \bullet f$.
- We prove the monotonicity lemma by induction on the structure of the sorting network (next slide).

Monotonicity Lemma (proof)

By induction. Base case:

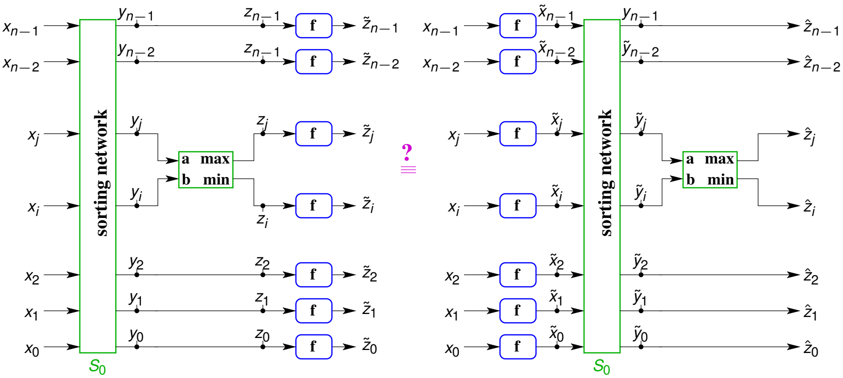


The sorting network, S, is the identity function.

$$f \bullet S \;=\; f \bullet \text{ident} \;=\; f \;=\; \text{ident} \bullet f \;=\; S \bullet f$$

Monotonicity Lemma (proof)

Induction step: Let S_0 be a sorting network, and append a compare-and-swap to outputs i and j.



Monotonicity Lemma (proof)

Definitions:

- S_0 is a sorting network, and
- $\mathrm{cas}_{i,j}$ is a compare-and-swap unit that compares the i^{th} and j^{th} outputs of S_0 to produce the i^{th} and j^{th} outputs of S.
- Without loss of generality, assume that the smaller value is output to the i^{th} output of S.
- Let x denote any input vector to $S \bullet f$ (or $f \bullet S$).
- Let $y = S_0(x)$, $z = S(x)$, $\tilde{x} = f(x)$, $\tilde{y} = f(y)$, and $\tilde{z} = f(z)$, and $\hat{z} = \mathrm{cas}_{i,j}(S_0(f(x)))$.
- We need to show that $\hat{z} = \tilde{z}$.

Monotonicity Lemma (proof)

Induction step: show $\hat{z} = \tilde{z}$.

- For any $k \not\in \{i, j\}$,

$$
\begin{array}{rcll}
\hat{z}_k & = & (\mathrm{cas}_{i,j}(S_0(f(x))))_k, & \text{definition of } \hat{z} \\
 & = & (S_0(f(x)))_k, & \text{definition of } \mathrm{cas}_{i,j} \\
 & = & (f(S_0(x)))_k & \text{induction hypothesis} \\
 & = & \tilde{y}_k, & \text{definition of } \tilde{y} \\
 & = & \tilde{z}_k, & \text{definition of } \mathrm{cas}_{i,j}
\end{array}
$$

- The i^{th} output:

$$
\begin{array}{rcll}
\hat{z}_i & = & (\mathrm{cas}_{i,j}(S_0(f(x))))_k, & \text{definition of } \hat{z} \\
 & = & \min((S_0(f(x)))_i, (S_0(f(x)))_j), & \text{definition of } \mathrm{cas}_{i,j} \\
 & = & \min((f(S_0(x))_i, (f(S_0(x)))_j), & \text{induction hypothesis} \\
 & = & f(\min((S_0(x)_i, (S_0(x))_j), & f \text{ is monotonic} \\
 & = & f(\mathrm{cas}_{i,j}((S_0(x))_i, (S_0(x))_j)), & \text{definition of } \mathrm{cas}_{i,j} \\
 & = & \tilde{z}_i, & \text{definition of } \tilde{z}
\end{array}
$$

- The j^{th} output: equivalent to the argument for the i^{th} output.

The 0-1 Principle

If a sorting network correctly sorts all inputs consisting only of 0's and 1's, then it correctly sorts inputs of any values.

I'll prove the contrapositive.

- If a sorting network does not correctly sort inputs of any values, then it does not correctly sort all inputs consisting only of 0's and 1's.
- Let S be a sorting network, let x be an input vector, and let $y = S(x)$, such that there exist i and j with $i < j$ such that $y_i > y_j$.
- Let
$$\begin{aligned} f(x) &= 0, && \text{if } x < y_i \\ &= 1, && \text{if } x \geq y_i \\ \tilde{y} &= S(f(x)) \end{aligned}$$
- By the definition of f, $f(x)$ is an input consisting only of 0's and 1's.
- By the monotonicity lemma, $\tilde{y} = f(y)$. Thus,
$$\tilde{y}_i = f(y_i) = 1 > 0 = f(y_j) = \tilde{y}_j$$
- Therefore, S does not correctly sort an input consisting only of 0's and 1's.
- $\square$

Announcements and reminders

- Nov. 20: Review Lin & Snyder, Chapter 5, *Scalable Parallelism* (the Bitonic Sort example).
  Read Lin & Snyder Chapter 6: *Programming with Threads*
- Nov. 22–29: Parallel computing examples

Review

- Why don't traditional, sequential sorting algorithms parallelize well?
- Try to parallelize another sequential sorting algorithm such as heap sort? What issues do you encounter?
- We proved that 0-1 principle for sorting networks. Show that the 0-1 principle does **not** apply to arbitrary programs. In particular, show a simple program (sequential is fine) that sorts all inputs of 0's and 1's correctly, but does not sort arbitrary inputs correctly.