Matrix Multiplication

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Outline:

- Sequential Matrix Multiplication
- Parallel Implementations, Performance, and Trade-Offs.

Sequential Matrix Multiplication

From September 11 & 13 slides:

```
mult(A, B) ->
   BT = transpose(B),
   lists:map(
      fun(RA) ->
         lists:map(
            fun(CB) -> dot_prod(RA, CB) end, BT)
      end, A).

dot_prod(V1, V2) ->
   lists:foldl(
      fun({X,Y},Sum) -> Sum + X*Y end,
      0, lists:zip(V1, V2)).
```

Now, we'll introduce List Comprehensions to get a more succinct version.

List Comprehensions

- Basic version: `[ Expr || X <- List` *, etc.* `]`
  - `Expr` is evaluated for each element, `X`, of `List`, to produce a list.
  - Example:

```
1> [ X*X || X <- lists:seq(1, 5) ].
[1,4,9,16,25]
```

- A list comprehension can apply to multiple lists:
  - Example:

```
2> [ X*X + Y || X <- lists:seq(1, 5), Y <- [1, 2] ].
[2,3,5,6,10,11,17,18,26,27].
```

  - Note the nesting:

    `for each` *First_Comprehension_Variable*
        `for each` *Second_Comprehension_Variable*
            `Expr`

- A list comprehension can have filters
  - Example:

```
3> [ X*X || X <- lists:seq(1, 5), (X rem 2) == 1].
[1,9,25]
```

Matrix Multiplication, with comprehensions

```
mult(A, B) ->
    BT = transpose(B),
    [ [ dot_prod(RowA, ColB) || ColB <- BT ] || RowA <- A].
```

List Comprehensions, practice examples

Use list comprehensions to implement erlang functions for the following problems:

- `double(List) -> lists:map(fun(X) 2*X end, List).`
  Replace `lists:map` with an erlang expression that uses a comprehension.
- `divisible(K, List) -> lists:filter(fun(N) (N rem K) == 0 end, List).`
  Returns the list of all elements of `List` that are divisible by `K`.
- `qsort([]) -> []; qsort(List1) -> List2.`
  Use two list comprehensions, one to find the elements of `List1` that are less than or equal to `hd(List1)`, and another to find the elements that are greater than `hd(List1)`. Sort these two with recursive calls to `qsort` and concatenate the results using `++`.

List Comprehensions, one more practice problem

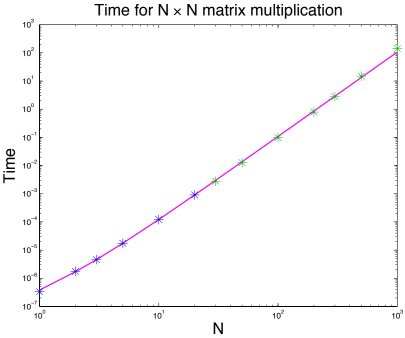
`pythag(ListX, ListY) -> ListP.`
`ListX` and `ListY` are lists of integers. `ListP` consists of all tuples `{X, Y}` `Y` is an element of `ListY`, and $\sqrt{\mathtt{X}^2 + \mathtt{Y}^2}$ is an integer. where `X` is an element of `ListX`, `Y` is an element of `ListY`, and $\mathtt{X} \leq \mathtt{Y}$ is an integer. Here's a function that tests whether or not an integer is a perfect square:

```
is_square(N, [Lo, Hi]) ->
   Mid = (Lo + Hi) div 2,
   MidSq = Mid*Mid,
   if
      (MidSq == N) -> true;
      (Lo >= Hi) -> false;
      (MidSq > N) -> is_square(N, [Lo, Mid]);
      (MidSq < N) -> is_square(N, [Mid+1, Hi])
   end.
```

Performance – Modeled

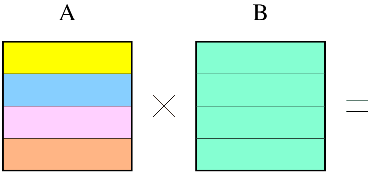
- Really simple, operation counts:
  - Multiplications: `n_rows_a` $*$ `n_cols_b` $*$ `n_cols_a`.
  - Additions: `n_rows_a` $*$ `n_cols_b` $*$ (`n_cols_a` $- 1$).
  - Memory-reads: 2$*$#Multiplications.
  - Memory-writes: `n_rows_a` $*$ `n_cols_b`.
  - Time is $O($`n_rows_a` $*$ `n_cols_b` $*$ (`n_cols_a` $- 1))$,
    If both matrices are $N \times N$, then its $O(N^3)$.
- But, memory access can be terrible.
  - For example, let matrices `a` and `b` be 1000×1000.
  - Assume a processor with a 4M L2-cache (final cache), 32 byte-cache lines, and a 200 cycle stall for main memory accesses.
  - Observe that a row of matrix `a` and a column of `b` fit in the cache. (a total of $\sim$40K bytes).
  - But, all of `b` does not fit in the cache (that's 8 Mbytes).
  - So, on every fourth pass through the inner loop, every read from `b` is a cache miss!
  - The cache miss time would dominate everything else.
- This is why there are carefully tuned numerical libraries.

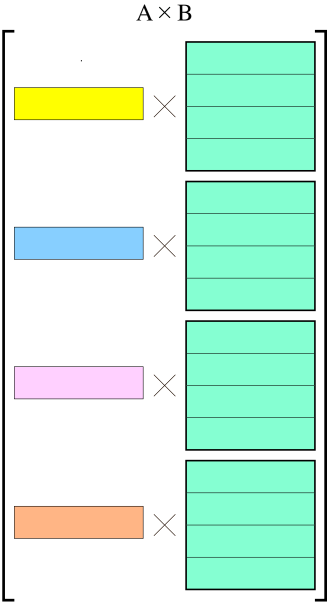
Performance – Measured



- Cubic of best fit: $T = (107N^3 + 134N^2 + 173N - 32)\mathrm{ns}$.
- Fit to first six data points.
- Cache misses effects are visible, for N=1000:
  - model predicts $T = 107\mathrm{seconds}$,
  - but the measured value is $T = 142\mathrm{seconds}$.

Parallel Algorithm 1





- Parallelize the outer-loop.
- Each iteration of the outer-loop multiplies a row of A by all of B to produce a row of $A \times B$.
- Divide A (and B) into blocks.
- Each processor sends its blocks of B to all of the the other processors.
- Now, each processor has a block of rows of A and all of B. The processor computes it's part of the product to produce a block of rows of C.
- Note: OpenMP does this kind of parallelization automatically.

Algorithm 1 in Erlang

```
par_matrix_mult1(ProcList, MyIndex, MyBlockA, MyBlockB) ->
    NProcs = length(ProcList),
    % send MyBlock to all other processes
    [ P ! {MyIndex, MyBlockB} || P <- ProcList],
    % receive all the blocks
    Bblocks = [    receive {I, Block} -> Block end
                || I <- lists:seq(1,NProcs) ],
    % concatenate these blocks to make the B matrix
    B = lists:append(Bblocks),
    matrix:mult(MyBlockA, B). % our block of A*B
```

The math:

- Let $A(i,:)$ denote the i^{th} row of A, and
- Let $B(:,j)$ denote the j^{th} column of B.
- Let $C = A * B$ we have: $C(i,:) \;=\; A(i,:) * B$.
- In English:
  - The processor that holds a block of rows of A can compute the corresponding rows of C.
  - The processor has to have all of B. That's what the sends and receives do at the begining of `par_matrix_mult1`.

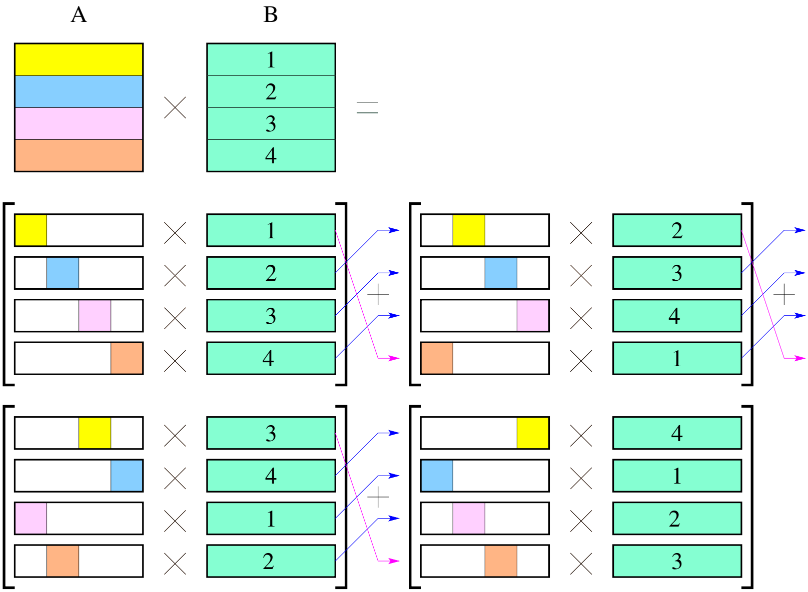
Performance of Parallel Algorithm 1

- CPU operations: same total number of multiplies and adds, but distributed around P processors. Total time: $O(N^3/P)$.
- Communication: Each processors sends (and receives) $P-1$ messages of size N^2/P. If time to send a message is $t_0 + t_1 * M$ where M is the size of the message, then the communication time is

$$\begin{aligned} (P-1)\left(t_0 + t_1 \frac{N^2}{P}\right) &= O(N^2+P), \quad \text{but, beware of large constants} \\ &= O(N^2), \qquad N^2 > P \end{aligned}$$

- Memory: Each process needs $O(N^2/P)$ storage for its block of A and the result. It also needs $O(N^2)$ to hold all of B.
  - The simple algorithm divides the computation across all processors, but it doesn't make good use of their combined memory.

Parallel Algorithm 2 (illustrated)



Parallel Algorithm 2 (code sketch)

- Each processor first computes what it can with its rows from A and B.
  - It can only use N/P of its columns of its block from A.
  - It uses its entire block from B.
  - We've now computed one of P matrices, where the sum of all of these matrices is the matrix AB.
- We view the processors as being arranged in a ring,
  - Each processor forwards its block of B to the next processor in the ring.
  - Each processor computes an new partial product of AB and adds it to what it had from the previous step.
  - This process continues until every block of B has been used by every processor.

Algorithm 2, Erlang

```
par_matrix_mult2(ProcList, MyIndex, MyBlockA, MyBlockB) ->
   NProcs = length(ProcList),
   NRowsA = length(A),
   NColsB = length(hd(B)), % assume length(B) > 0
   ABlocks0 = rotate(MyIndex, blockify_cols(A, NProcs)),
   PList = rotate(NProcs - (MyIndex-1),
                  lists:reverse(ProcList)),
   helper(ProcList, ABlocks, MyBlockB,
          matrix:zeros(NRowsA, NColsB)).

helper([P_head | P_tail], [A_head | A_tail], BBlock, Accum) ->
   if A_tail == [] -> ok;
      true         -> P_head !  BBlock
   end,
   Accum2 = matrix:add(Accum, matrix:mult(A_head, BBlock)),
   if A_tail == [] -> Accum2;
      true ->
         helper(P_tail, A_tail,
                receive BBlock2 -> BBlock2 end, Accum2)
   end.
```

Algorithm 2 – notes on the Erlang code

- `blockify_cols(A, NProcs)` produces a list of `NProcs` matrices.
  - Each matrix has `NRowsA` rows and `NColsA` columns,
  - where `NColsA` is the number of columns of `MyBlockA`.
  - Let $A(\mathtt{MyIndex}, j)$ denote the j^{th} such block.
-
```
rotate(N, List) ->
    {L1, L2} = lists:split(N, List),
    L2 ++ L1.
```
- The algorithm is based on the formula:

$$C(\mathtt{MyIndex}, :) \;=\; \sum_{j=1}^{\mathtt{NProcs}} A(\mathtt{MyIndex}, j) * B(j, :)$$

Performance of Parallel Algorithm 2

- CPU operations: Same as for parallel algorithm 1: total time: $O(N^3/P)$.
- Communication: Same as for parallel algorithm 1: $O(N^2 + P)$.
  - With algorithm 1, each processor sent the same message to $P - 1$ different processors.
  - With algorithm 2, for each processor, there is one destination to which it sends $P - 1$ different messages.
  - Thus, algorithm 2 can work efficiently with simpler interconnect networks.
- Memory: Each process needs $O(N^2/P)$ storage for its block of A, its current block of B, and its block of the result.
  - Note: each processor might hold onto its original block of B so we still have the blocks of B available at the expected processors for future operations.
- Do the memory savings matter?

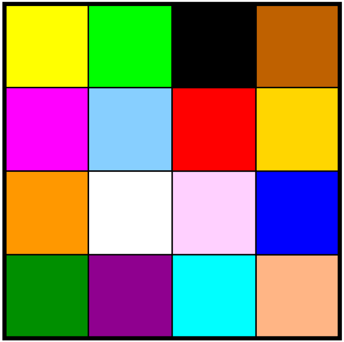
Bad performance, pass it on

- Consider what happens with algorithm 2 if one processor, P_{slow} takes a bit longer than the others one of the times its doing a block multiply.
  - P_{slow} will send it's block from B to its neighbour a bit later than it would have otherwise.
  - Even if the neighbour had finished its previous computation on time, it won't be able to start the next one until it gets the block of B from P_{slow}.
  - Thus, for the next block computation, both P_{slow} and its neighbour will be late, even if both of them do their next block computation in the usual time.
  - In other words, tardiness propagates.
- Solution: forward your block to you neighbour before you use it to perform a block computation.
  - This overlaps computation with communication, generally a good idea.
  - We could send two or more blocks ahead if needed to compensate for communication delays and variation in compute times.
  - This is a way to save time by using more memory.

Even less communication

- In the previous algorithms, computate time grows as N^3/P, while communication time goes as $(N^2 + P)$.
- Thus, if N is big enough, computation time will dominate communication time.
- There's not much we can do to reduce the number of computations required (I'll ignore Strassen's algorithm, etc. for simplicity).
- If we can use less communication, then we'll we won't need our matrices to be as huge to benefit from parallel computation.

Other ways to distribute a matrix



Lower bound for communication