Student’s Name: ________________________________

(Please print in BLOCK letters, SURNAME first.)

Student Number: ________________________________

Signature: ________________________________

Instructor’s Name: Rob Scharein

This examination consists of 8 pages, including this cover page.

Check to ensure that this exam is complete.

This is a closed book examination.

The weight of each question is given in parentheses. The total number of marks is 50. Start with
the questions you think are easiest, and then go back and do the harder ones. Show all your work.

Good Luck!

1. Each candidate should be prepared to produce, upon request, his/her library card.

2. READ AND OBSERVE THE FOLLOWING RULES:

   ● No candidate shall be permitted to ask questions of the invigilators, except in the cases
     of supposed errors or ambiguities in examination questions.

   ● CAUTION – Candidates suspected of any of the following, or similar, dishonest prac-
     tices shall be immediately dismissed from the examination and shall be liable to disci-
     plinary action.

     (a) Having at the place of writing communication devices, any books, papers or mem-
         oranda, audio or video players, FAX machines, mobile radio telephones, electronic
         computational devices, microfiche readers, or other memory aid devices other than
         that specifically approved by the instructor.

     (b) Speaking or communicating with other candidates.

     (c) Purposely exposing written papers to the view of other candidates. The plea of
         accident or forgetfulness shall not be received.

   ● The burning of any material is not permitted during examinations.

<table>
<thead>
<tr>
<th></th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
<th>Q5</th>
<th>Q6</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
<td>10</td>
<td>5</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>50</td>
</tr>
</tbody>
</table>
Question 1  (10 marks)

Given the OpenGL fragment shown on the right, together with the notation
\[ S(a, b, c) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T(r, s, t) = \begin{pmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

for scalings and translations, and
\[ R_x(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta & 0 \\ 0 & \sin \beta & \cos \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

for rotations about the x-axis (and similar equations for \( R_y(\beta) \) and \( R_z(\beta) \), the rotations about the other coordinate axes), write down the value for the matrix \( A \) in terms of products of the matrices \( B, S, T, R_x, R_y \), and \( R_z \) (you don’t have to work out the matrix products).

Answer:

<table>
<thead>
<tr>
<th>After call to line</th>
<th>Modelview matrix is</th>
<th>Matrix on stack is</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Identity</td>
<td>—</td>
</tr>
<tr>
<td>2</td>
<td>( T(8, 2, -4) )</td>
<td>—</td>
</tr>
<tr>
<td>3</td>
<td>( T(8, 2, -4) )</td>
<td>( T(8, 2, -4) )</td>
</tr>
<tr>
<td>4</td>
<td>( T(8, 2, -4) ) ( S(2, 3, 1) )</td>
<td>( T(8, 2, -4) )</td>
</tr>
<tr>
<td>5</td>
<td>( T(8, 2, -4) ) ( S(2, 3, 1) B )</td>
<td>( T(8, 2, -4) )</td>
</tr>
<tr>
<td>6</td>
<td>( C = T(8, 2, -4) ) ( S(2, 3, 1) B )</td>
<td>( T(8, 2, -4) )</td>
</tr>
<tr>
<td>7</td>
<td>( T(8, 2, -4) )</td>
<td>—</td>
</tr>
<tr>
<td>8</td>
<td>( T(8, 2, -4) ) ( R_y(40) )</td>
<td>—</td>
</tr>
<tr>
<td>9</td>
<td>( T(8, 2, -4) ) ( R_y(40) ) ( S(2, -1, 3) )</td>
<td>—</td>
</tr>
<tr>
<td>10</td>
<td>( T(8, 2, -4) ) ( R_y(40) ) ( S(2, -1, 3) C )</td>
<td>—</td>
</tr>
<tr>
<td>11</td>
<td>( T(8, 2, -4) ) ( R_y(40) ) ( S(2, -1, 3) C R_x(-120) )</td>
<td>—</td>
</tr>
<tr>
<td>12</td>
<td>( A = T(8, 2, -4) ) ( R_y(40) ) ( S(2, -1, 3) C R_x(-120) )</td>
<td>—</td>
</tr>
</tbody>
</table>

Substituting for \( C \), we get

\[ A = T(8, 2, -4) \) \( R_y(40) \) \( S(2, -1, 3) \) \( T(8, 2, -4) \) \( S(2, 3, 1) B R_x(-120) \) \]
Question 2  (10 marks)

Given a conical helix defined by the parametric equations

\[ C(t) = (t \cos t, t \sin t, t), \]

compute the Frenet frame \((T, N, B)\) along the curve at the parameter value \(t = 0\). Is the second derivative of the path \(C''(t)\) perpendicular to \(T\) at \(t = 0\)? If not, what is the angle between the two vectors?

Answer:

In the following, \(T\), \(N\), and \(B\) are the Frenet frame at \(t = 0\).

\[
\begin{align*}
C'(t) &= (\cos t - t \sin t, \sin t + t \cos t, 1) \\
C'(0) &= (1, 0, 1) \\
C''(t) &= (-\sin t - \sin t - t \cos t, \cos t + \cos t - t \sin t, 0) \\
C''(0) &= (-2 \sin t - t \cos t, 2 \cos t - t \sin t, 0) \\
C''(0) &= (0, 2, 0)
\end{align*}
\]

\[
T = \frac{C'(0)}{|C'(0)|} = \frac{1}{\sqrt{2}} (1, 0, 1)
\]

\[
B = \frac{C'(0) \times C''(0)}{|C'(0) \times C''(0)|} \\
= \frac{(1, 0, 1) \times (0, 2, 0)}{|(1, 0, 1) \times (0, 2, 0)|} \\
= \frac{(-2, 0, 2)}{|(-2, 0, 2)|} \\
B = \frac{1}{\sqrt{2}} (-1, 0, 1)
\]

\[
N = B \times T \\
= \frac{1}{2} (-1, 0, 1) \times (1, 0, 1) \\
= \frac{1}{2} (0, 2, 0)
\]

Since \(C''(0) \cdot T = (0, 2, 0) \cdot \frac{1}{\sqrt{2}} (1, 0, 1) = 0\), \(C''(0)\) and \(T\) are perpendicular.
**Question 3  (5 marks)**

Recall that two methods were used in Assignment #2 to create extruded surfaces, one method used the Frenet frame, and the other method used the parallel transport frame. Discuss the advantages and disadvantages of each method.

<table>
<thead>
<tr>
<th>Advantages</th>
<th>Frenet</th>
<th>Parallel transport</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple to compute at any point.</td>
<td>Frame rotates as little as possible about tangent vector, therefore good for creating “nice” extrusions.</td>
<td></td>
</tr>
<tr>
<td>If defined, frame is unique.</td>
<td>Defined everywhere, even for straight lines.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Disadvantages</th>
<th>Frenet</th>
<th>Parallel transport</th>
</tr>
</thead>
<tbody>
<tr>
<td>May not be defined (e.g., curve is straight).</td>
<td>For closed curves, frame may not match at beginning and end (there is an extra twist).</td>
<td></td>
</tr>
<tr>
<td>Can have sudden twists which results in poorly formed extruded surfaces.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Question 4  (5 marks)**

What does affine invariance of a Bézier curve mean? Explain why Bézier curves (of any order) are affine invariant.

**Answer:**

Given an arbitrary affine map, \( A \), and control points for a Bézier curve, \( P_i \), affine invariance means that the following two procedures result in the same curve:

- first draw the Bézier curve based on the control points \( P_i \) and then apply the map \( A \) to the curve
- first apply the map \( A \) to the points \( P_i \) and then draw the Bézier curve based on the transformed points

Reason why: Bézier curves are formed as an affine combination of points. Affine transformations preserve affine combinations.
Question 5  (10 marks)

Given four points $P_0, P_1, P_2,$ and $P_3$, the cubic Bézier curve that interpolates $P_0$ and $P_3$ can be written as

$$P(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_1 + 3(1-t)t^2 P_2 + t^3 P_3, \text{ for } 0 \leq t \leq 1.$$ 

Choose any assignment of $P_0, P_1, P_2,$ and $P_3$ to the four points shown below (as long as the assignment is 1-to-1) and show how you would construct $P(1/3)$ using the de Casteljau algorithm. Label the points clearly and show all your work. In addition to finding $P(1/3)$ precisely, sketch an approximation to the complete Bézier curve.

**Answer:**

This answer is for the most common assignment of labels to the points.

$P_{0121223}$ is the point $P(1/3)$. 

Diagram:

- $P_0$,
- $P_1$,
- $P_2$,
- $P_3$,
- $P_{012}$,
- $P_{0112}$,
- $P_{01123}$,
- $P_{01223}$,
- $P_{011223}$,
- $P_{12}$,
- $P_{1223}$,
- $P_{0121223}$.
Question 6 (10 marks)

Consider the paths
\[ \gamma(t) = (t, t^2) \quad \text{and} \quad \eta(t) = (2t + 1, t^3 + 4t + 1), \]
both defined on the interval \(0 \leq t \leq 1\). The curves join, since \(\gamma(1) = (1, 1) = \eta(0)\). Show that they meet with \(G^1\) continuity, but not with \(C^1\) continuity.

**Answer:**
First calculate the derivatives
\[ \gamma'(t) = (1, 2t) \]
\[ \eta'(t) = (2, 3t^2 + 4) \]
and then substitute values for the endpoints,
\[ \gamma'(1) = (1, 2) \]
\[ \eta'(0) = (2, 4) \]

Since \(2\gamma'(1) = \eta'(0)\) the curve is \(G^1\) at the join.
Since \(\gamma'(1) \neq \eta'(0)\) the curve is not \(C^1\) at the join.
Extra space in which to work
More extra space in which to work