

CPSC 340: Machine Learning and Data Mining

Decision Trees

Admin

- **Assignment 1** is due next Friday: finish early!
- Waiting list people
 - Assume it will work, do the assignment
- Tutorials and office hours start next week (webpage has the times)

Last Time: Data Representation and Exploration

- We discussed **example-feature representation**:

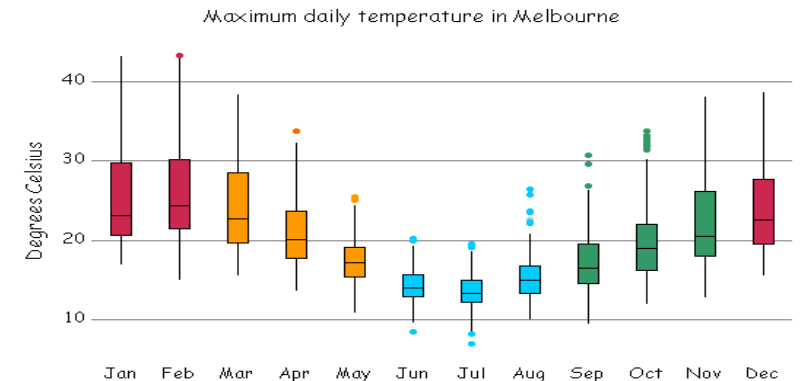
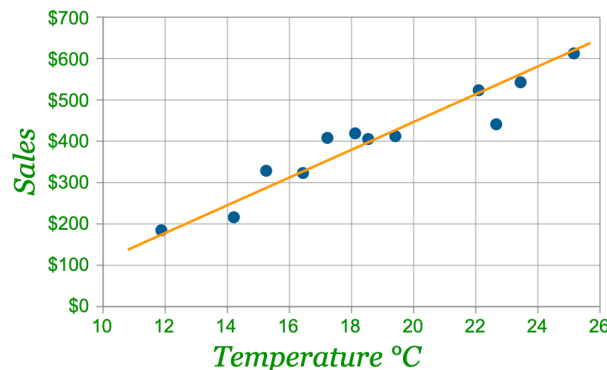
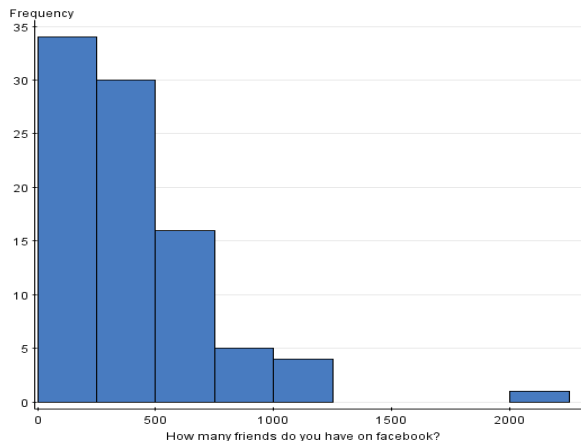
- Examples/samples/data points

| Age | Job? | City | Rating | Income |
|-----|------|------|--------|-----------|
| 23 | Yes | Van | A | 22,000.00 |
| 23 | Yes | Bur | BBB | 21,000.00 |
| 22 | No | Van | CC | 0.00 |
| 25 | Yes | Sur | AAA | 57,000.00 |

"Feature"

"Example"

- We discussed **summary statistics and visualizing data**.



Last Time: Supervised Learning

- We discussed supervised learning:

| Egg | Milk | Fish | Wheat | Shellfish | Peanuts | ... | Sick? |
|-----|------|------|-------|-----------|---------|-----|-------|
| 0 | 0.7 | 0 | 0.3 | 0 | 0 | | 1 |
| 0.3 | 0.7 | 0 | 0.6 | 0 | 0.01 | | 1 |
| 0 | 0 | 0 | 0.8 | 0 | 0 | | 0 |
| 0.3 | 0.7 | 1.2 | 0 | 0.10 | 0.01 | | 1 |
| 0.3 | 0 | 1.2 | 0.3 | 0.10 | 0.01 | | 1 |

- Input for an example (day of the week) is a set of features (quantities of food).
- Output is a desired class label (whether or not we got sick).
- Goal of supervised learning:
 - Use data to find a model that outputs the right label based on the features.
 - Above, model predicts whether foods will make you sick (even with new combinations).
 - This framework can be applied any problem where we have input/output examples.

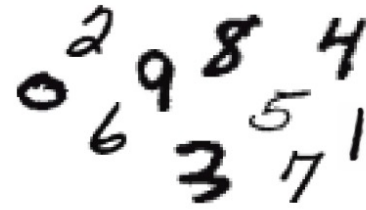
Examples of Supervised Learning

Document Classification



{no spam, spam}

Digit Classification



{0,1,2,3,4,5,6,7,8,9}

Face Detection



{no face, face}

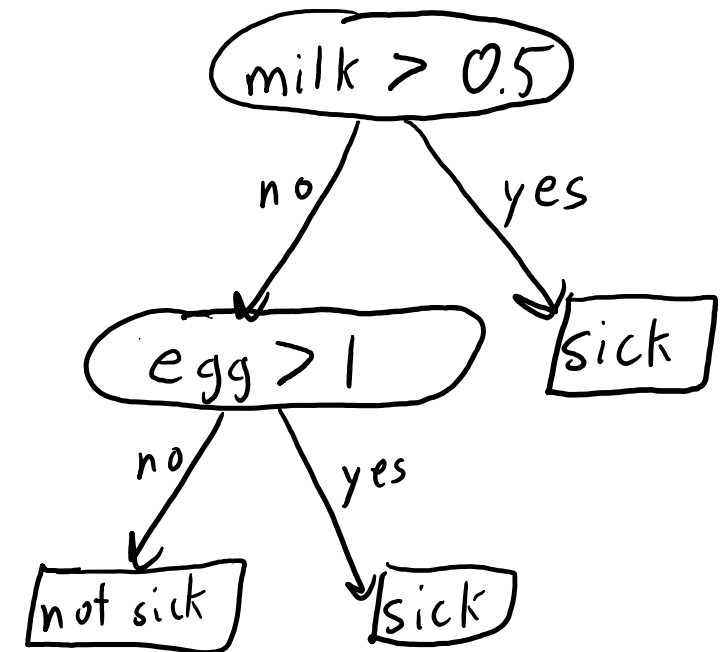
Decision Trees

- **Decision trees** are simple programs consisting of:
 - A nested sequence of “if-else” decisions based on the features (**splitting rules**).
 - A **class label as a return value** at the end of each sequence.

- Example **decision tree**:

```
1  if milk > 0.5:  
2      return 'sick'  
3  else:  
4      if egg > 1:  
5          return 'sick'  
6      else:  
7          return 'not sick'
```

Can draw sequences of **decisions as a tree**:



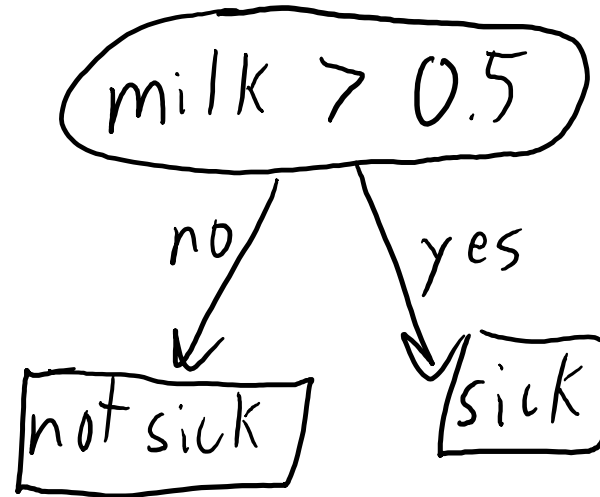
| Egg | Milk | Fish | Wheat | Shellfish | Peanuts | Sick ? |
|-----|------|------|-------|-----------|---------|--------|
| 0.8 | 0.2 | 0 | 0.3 | 0 | 0 | ? |

Supervised Learning as Writing A Program

- There are many possible decision trees.
 - We're going to search for one that is good at our supervised learning problem.
- So our input is data and the output will be a program.
 - This is called "training" the supervised learning model.
 - Different than usual input/output specification for writing a program.
- Supervised learning is useful when you have lots of labeled data BUT:
 - Problem is too complicated to write a program ourselves.
 - Human expert can't explain why you assign certain labels.
 - We don't have a human expert for the problem.
 - Or we want to automate labor

Learning A Decision Stump: “Search and Score”

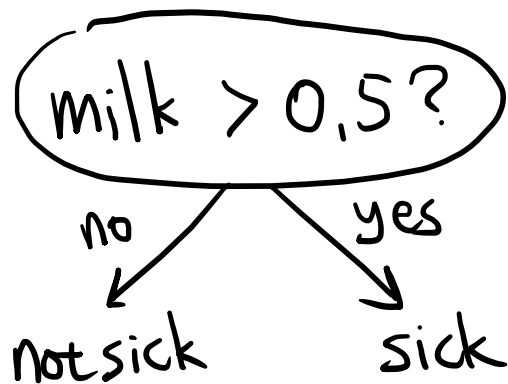
- We’ll start with “**decision stumps**”:
 - Simple decision tree with **1 splitting rule based on thresholding 1 feature**.



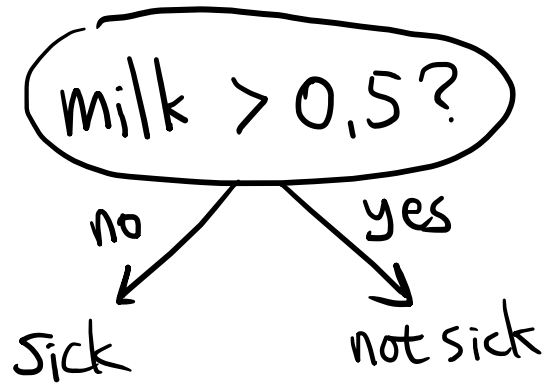
- To “learn” a decision stump we need to find 3 things:
 - Which feature should we use to “split” the data?
 - What value of the threshold should be used?
 - What classes should we use at the leaves?

Learning A Decision Stump: “Search and Score”

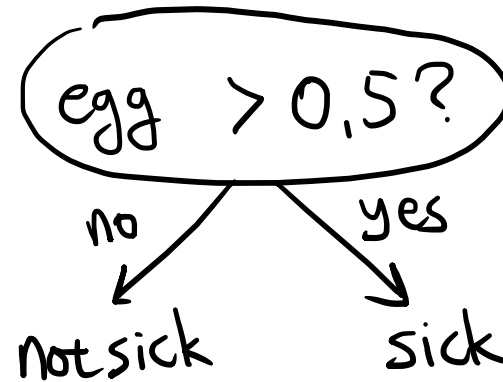
- To “learn” a decision stump:
 1. Define a ‘score’ for each possible rule.
 2. Search for the rule with the best score.



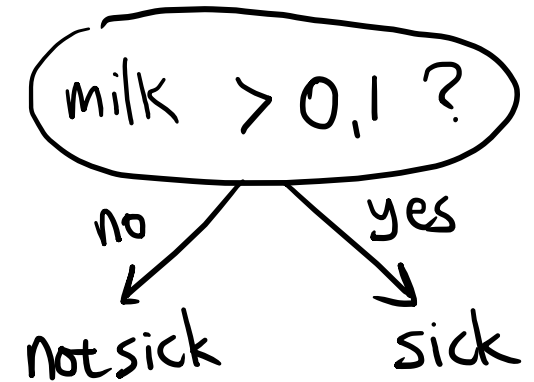
Score=50



Score=10



Score=45



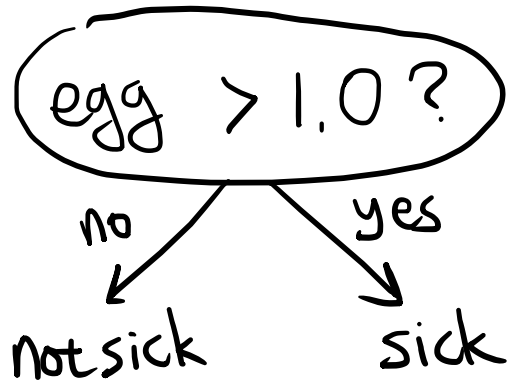
Score=65

(Best rule!)

- Q: what “score” should we use?

Learning A Decision Stump: Accuracy Score

- Most intuitive score: **classification accuracy**.
 - “If we use this rule, how many examples do we label correctly?”



| Milk | Fish | Egg | Sick? |
|------|------|-----|-------|
| 0.7 | 0 | 1 | 1 |
| 0.7 | 0 | 2 | 1 |
| 0 | 0 | 0 | 0 |
| 0.7 | 1.2 | 0 | 0 |
| 0 | 1.2 | 2 | 1 |
| 0 | 0 | 0 | 0 |

- Computing classification accuracy for this rule:
 - Find **most common labels** after applying splitting rule:
 - When $(egg > 1)$, we were “sick” 2 times out of 2.
 - When $(egg \leq 1)$, we were “not sick” 3 times out of 4.
 - Compute **accuracy**:
 - The accuracy (“score”) of the rule is **5 times out of 6**.
- This “**score**” evaluates quality of a rule.
 - We “learn” a decision stump by **finding the rule with the best score**.

Learning A Decision Stump: By Hand

- Let's **search** for the decision stump maximizing classification **score**:

| Milk | Fish | Egg | Sick? |
|------|------|-----|-------|
| 0.7 | 0 | 1 | 1 |
| 0.7 | 0 | 2 | 1 |
| 0 | 1.2 | 0 | 0 |
| 0.7 | 1.2 | 0 | 0 |
| 0 | 1.3 | 2 | 1 |
| 0 | 0 | 0 | 0 |

First we check “**baseline rule**” of predicting mode (no split): this gets 3/6 accuracy.

If (milk > 0) predict “sick” (2/3) else predict “not sick” (2/3): 4/6 accuracy

If (fish > 0) predict “not sick” (2/3) else predict “sick” (2/3): 4/6 accuracy

If (fish > 1.2) predict “sick” (1/1) else predict “not sick” (3/5): 4/6 accuracy

If (egg > 0) predict “sick” (3/3) else predict “not sick” (3/3): 6/6 accuracy

If (egg > 1) predict “sick” (2/2) else predict “not sick” (3/4): 5/6 accuracy

- Highest-scoring rule**: (egg > 0) with leaves “sick” and “not sick”.
- Notice we **only need to test feature thresholds that happen** in the data:
 - There is no point in testing the rule (egg > 3), it gets the “baseline” score.
 - There is no point in testing the rule (egg > 0.5), it gets the (egg > 0) score.
 - Also note that we don't need to test “<”, since it would give equivalent rules.

Supervised Learning Notation (MEMORIZE THIS)

$X =$

| Egg | Milk | Fish | Wheat | Shellfish | Peanuts |
|-----|------|------|-------|-----------|---------|
| 0 | 0.7 | 0 | 0.3 | 0 | 0 |
| 0.3 | 0.7 | 0 | 0.6 | 0 | 0.01 |
| 0 | 0 | 0 | 0.8 | 0 | 0 |
| 0.3 | 0.7 | 1.2 | 0 | 0.10 | 0.01 |
| 0.3 | 0 | 1.2 | 0.3 | 0.10 | 0.01 |

$y =$

| Sick? |
|-------|
| 1 |
| 1 |
| 0 |
| 1 |
| 1 |

Handwritten annotations: A red bracket on the right of the matrix X is labeled $'n'$. A red bracket on the right of the vector y is labeled $'n'$. A red bracket under the bottom row of X is labeled $'d'$.

- Feature matrix $'X'$ has rows as examples, columns as features.
 - x_{ij} is feature $'j'$ for example $'i'$ (quantity of food $'j'$ on day $'i'$).
 - x_i is a list of all features for example $'i'$ (all the quantities on day $'i'$)
 - x^j is column $'j'$ of the matrix (the value of feature $'j'$ across all examples).
- Label vector $'y'$ contains the labels of the examples.
 - y_i is the label of example $'i'$ (1 for “sick”, 0 for “not sick”).

Note!
lowercase x

Note: a list (the data type), not a vector

Supervised Learning Notation (MEMORIZE THIS)

$X =$

| Egg | Milk | Fish | Wheat | Shellfish | Peanuts |
|-----|------|------|-------|-----------|---------|
| 0 | 0.7 | 0 | 0.3 | 0 | 0 |
| 0.3 | 0.7 | 0 | 0.6 | 0 | 0.01 |
| 0 | 0 | 0 | 0.8 | 0 | 0 |
| 0.3 | 0.7 | 1.2 | 0 | 0.10 | 0.01 |
| 0.3 | 0 | 1.2 | 0.3 | 0.10 | 0.01 |

$y =$

| Sick? |
|-------|
| 1 |
| 1 |
| 0 |
| 1 |
| 1 |

Handwritten annotations:

- Red curly braces on the right of the X and y matrices are labeled $'n'$.
- Red curly braces on the right of the y matrix are labeled $'h'$.
- Green circles highlight the values 0.10 in the Shellfish column of the 4th and 5th rows of X , and the value 1 in the 4th row of y .
- Green arrows point from the 0.10 values in X to the text $x_{45} = 0.10$ and $y_4 = 1$.
- Green arrows point from the 2nd and 3rd rows of X to the handwritten vectors $x_2 = (0.3, 0.7, 0, 0.6, 0, 0.01)$ and $x^3 = (0, 0, 0, 1.2, 1.2)$.
- The text "lowercase x's" is written below the X matrix.

Supervised Learning Notation (MEMORIZE THIS)

$X =$

| Egg | Milk | Fish | Wheat | Shellfish | Peanuts |
|-----|------|------|-------|-----------|---------|
| 0 | 0.7 | 0 | 0.3 | 0 | 0 |
| 0.3 | 0.7 | 0 | 0.6 | 0 | 0.01 |
| 0 | 0 | 0 | 0.8 | 0 | 0 |
| 0.3 | 0.7 | 1.2 | 0 | 0.10 | 0.01 |
| 0.3 | 0 | 1.2 | 0.3 | 0.10 | 0.01 |

$y =$

| Sick? |
|-------|
| 1 |
| 1 |
| 0 |
| 1 |
| 1 |

Handwritten annotations: A red bracket on the right of the X matrix is labeled n . A red bracket on the right of the y vector is labeled n . A red bracket at the bottom of the X matrix is labeled d .

- **Training phase:**
 - Use ‘ X ’ and ‘ y ’ to **find a ‘model’** (like a decision stump).
- **Prediction phase:**
 - Given an example x_i , use ‘model’ to **predict a label ‘ \hat{y}_i ’** (“sick” or “not sick”).
- **Training error:**
 - Fraction of **times our prediction \hat{y}_i does not equal the true y_i label.**

Decision Stump Learning Pseudo-Code

Input: feature matrix X and label vector y

Compute error if using "baseline" rule: number of times y_i does not equal most common value.
for each feature 'j' (column of 'X')

for each threshold 't'

set ' y_{yes} ' to most common label of objects 'i' satisfying rule ($x_{ij} > t$)

set ' y_{no} ' to most common label of objects not satisfying rule.

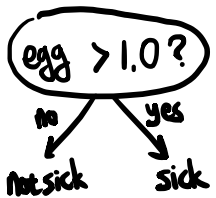
set ' \hat{y}_i ' to be our predictions for each object 'i' based on the rule.

compute error 'E', number of objects where $\hat{y}_i \neq y_i$ ($\hat{y}_i = y_{\text{yes}}$ if satisfied,
 $\hat{y}_i = y_{\text{no}}$ if not satisfied)

store the rule $(j, t, y_{\text{yes}}, y_{\text{no}})$ if it has the lowest error so far.

Output: a "best" decision stump based on score. (the "model")

Y=yes = set of things routed to the yes side



Cost of Decision Stumps

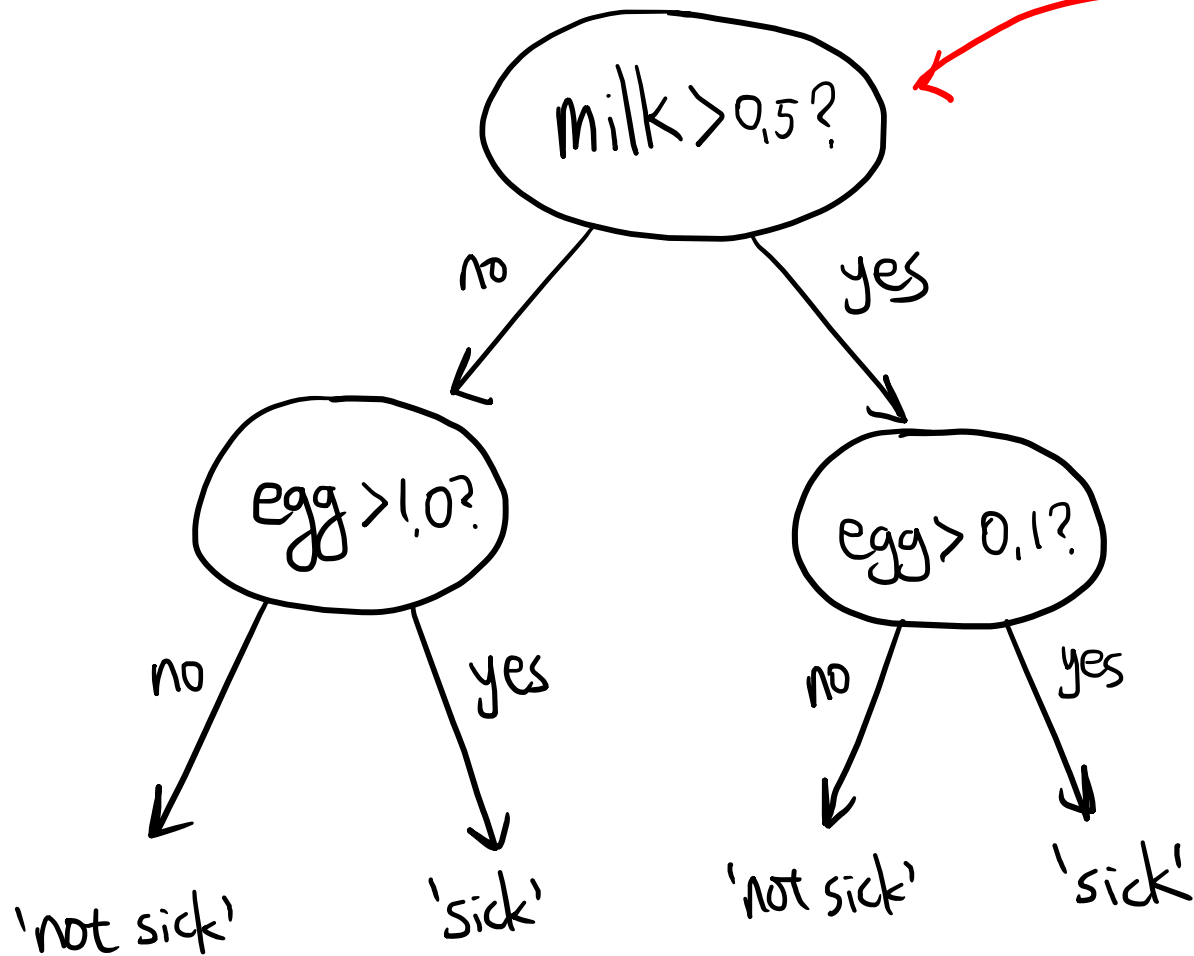
- How much does this cost?
- Assume we have:
 - ‘n’ examples (days that we measured).
 - ‘d’ features (foods that we measured).
 - ‘k’ thresholds (>0 , >1 , >2 , ...) for each feature.
- Computing the score of one rule costs $O(n)$:
 - We need to go through all ‘n’ examples to find most common labels.
 - We need to go through all ‘n’ examples again to compute the accuracy.
 - See notes on webpage for review of “ $O(n)$ ” notation.
- We compute score for up to $k*d$ rules (‘k’ thresholds for each of ‘d’ features):
 - So we need to do an $O(n)$ operation $k*d$ times, giving total cost of $O(ndk)$.

Cost of Decision Stumps

- Is a cost of $O(ndk)$ good?
- Size of the input data is $O(nd)$:
 - If 'k' is small then the cost is roughly the same cost as loading the data.
 - We should be happy about this, you can learn on any dataset you can load!
 - If 'k' is large then this could be too slow for large datasets.
- Example: if all our features are **binary** then $k=1$, just test (feature > 0):
 - Cost of fitting decision stump is $O(nd)$, so **we can fit huge datasets**.
- Example: if all our features are **numerical** with unique values then $k=n$.
 - Cost of fitting decision stump is $O(n^2d)$.
 - **We don't like having n^2** because we want to fit datasets where 'n' is large!
 - Bonus slides: how to **reduce the cost in this case down to $O(nd \log n)$** .
 - Basic idea: sort features and track labels. **Allows us to fit decision stumps to huge datasets**.

Next Topic: Learning Decision Trees

Decision Stumps and Decision Trees



```
7 def decision_tree():
8     if milk > 0.5:
9         if egg > 0.1:
10            return 'sick'
11        else:
12            return 'not sick'
13    else:
14        if egg > 1.0:
15            return 'sick'
16        else:
17            return 'not sick'
```

these are
decision stumps!

Decision Tree Learning

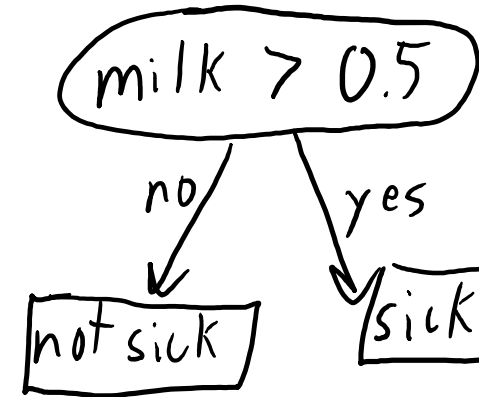
- **Decision stumps** have only 1 rule based on only 1 feature.
 - Very limited class of models: usually not very accurate for most tasks.
- **Decision trees** allow **sequences of splits** based on multiple features.
 - Very general class of models: can get very high accuracy.
 - However, it's **computationally infeasible to find the best decision tree**
 - for a given depth
- Most common decision tree learning algorithm in practice:
 - **Greedy recursive splitting.**

Example of Greedy Recursive Splitting

- Start with the full dataset:

| Egg | Milk | ... | Sick? |
|-----|------|-----|-------|
| 0 | 0.7 | | 1 |
| 1 | 0.7 | | 1 |
| 0 | 0.5 | | 0 |
| 1 | 0.6 | | 1 |
| 1 | 0 | | 0 |
| 2 | 0.6 | | 1 |
| 0 | 1 | | 1 |
| 2 | 0 | | 1 |
| 0 | 0.3 | | 0 |
| 1 | 0.6 | | 0 |
| 2 | 0 | | 1 |

Find the decision stump with the best score:



Split into **two smaller datasets** based on stump:

| Egg | Milk | ... | Sick? |
|-----|------|-----|-------|
| 0 | 0.5 | | 0 |
| 1 | 0 | | 0 |
| 2 | 0 | | 1 |
| 0 | 0.3 | | 0 |
| 2 | 0 | | 1 |

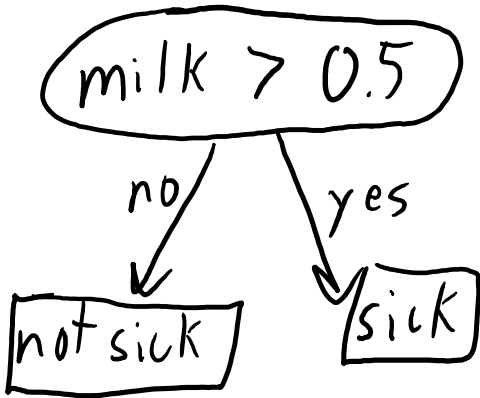
milk ≤ 0.5

| Egg | Milk | ... | Sick? |
|-----|------|-----|-------|
| 0 | 0.7 | | 1 |
| 1 | 0.7 | | 1 |
| 1 | 0.6 | | 1 |
| 2 | 0.6 | | 1 |
| 0 | 1 | | 1 |
| 1 | 0.6 | | 0 |

> 0.5

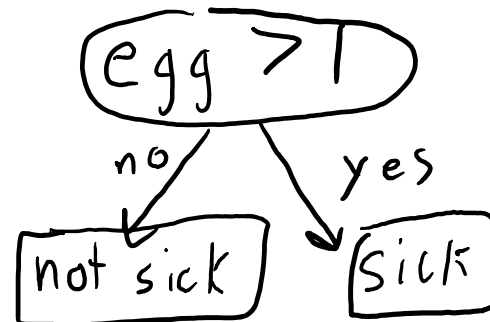
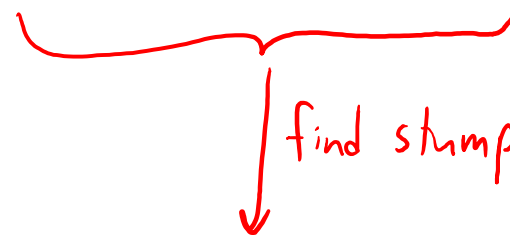
Greedy Recursive Splitting

We now have a decision stump and two datasets:

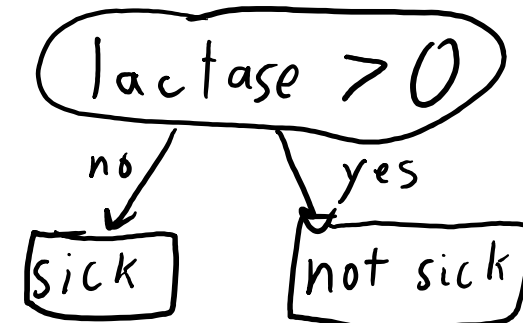
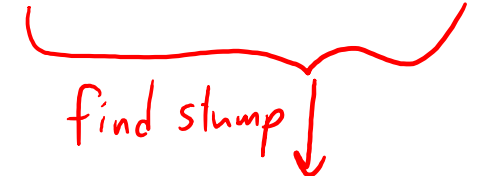


Fit a decision stump to each leaf's data.

| Egg | Milk | ... | Sick? |
|-----|------|-----|-------|
| 0 | 0.5 | | 0 |
| 1 | 0 | | 0 |
| 2 | 0 | | 1 |
| 0 | 0.3 | | 0 |
| 2 | 0 | | 1 |

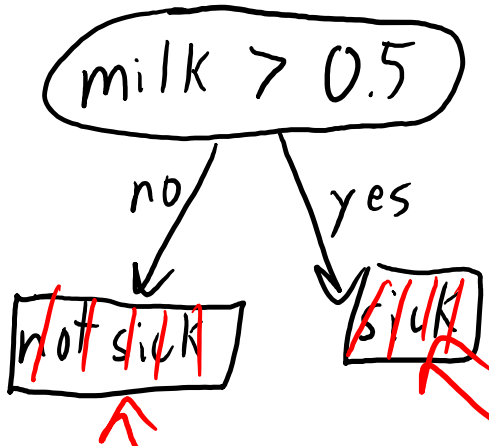


| Egg | Milk | ... | Sick? |
|-----|------|-----|-------|
| 0 | 0.7 | | 1 |
| 1 | 0.7 | | 1 |
| 1 | 0.6 | | 1 |
| 2 | 0.6 | | 1 |
| 0 | 1 | | 1 |
| 1 | 0.6 | | 0 |



Greedy Recursive Splitting

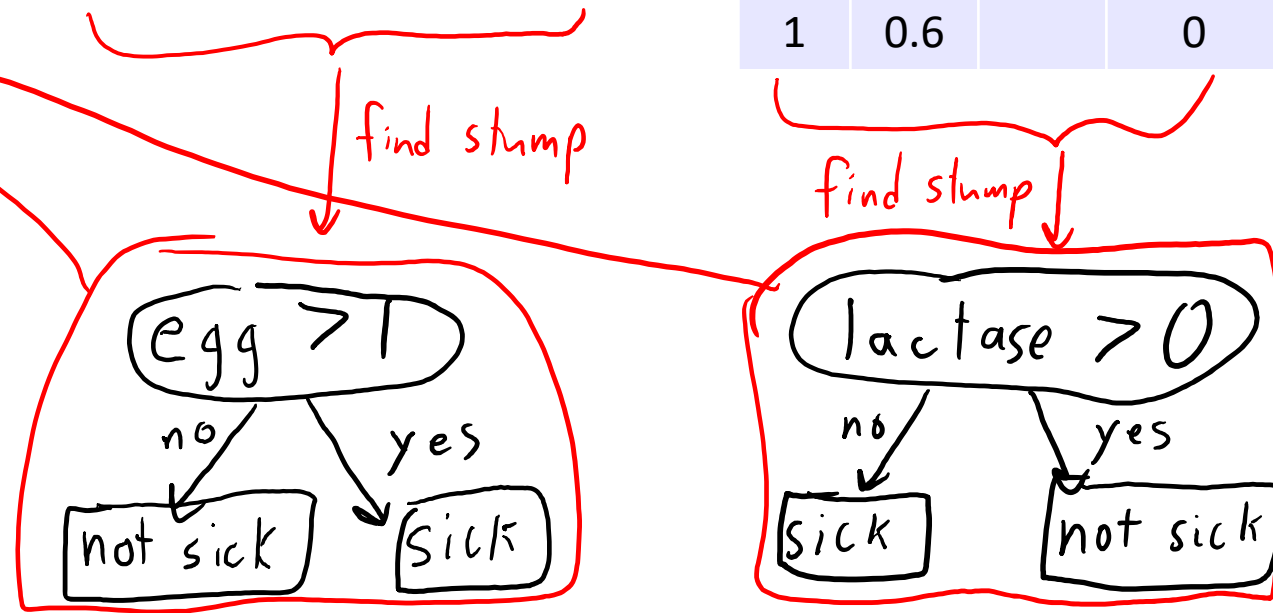
We now have a decision stump and two datasets:



| Egg | Milk | ... | Sick? |
|-----|------|-----|-------|
| 0 | 0.5 | | 0 |
| 1 | 0 | | 0 |
| 2 | 0 | | 1 |
| 0 | 0.3 | | 0 |
| 2 | 0 | | 1 |

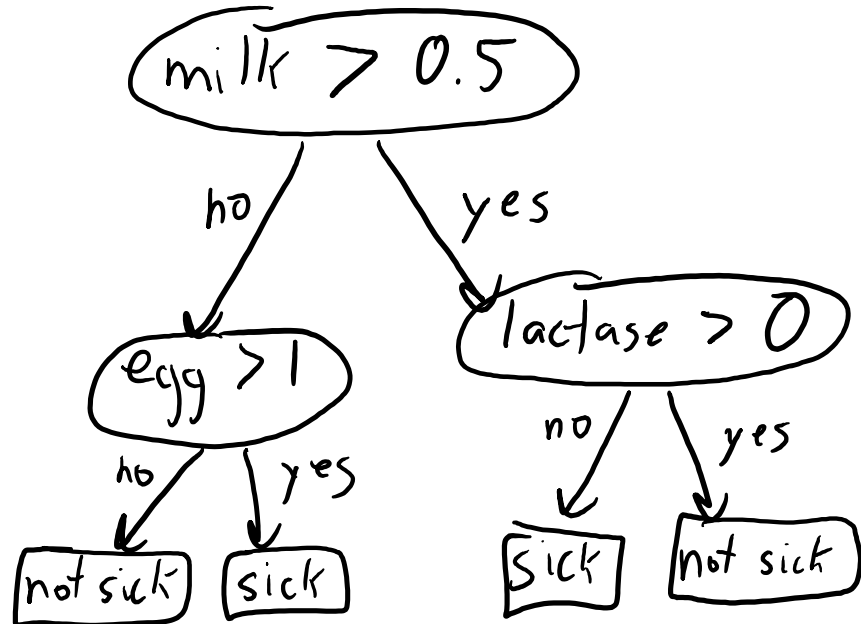
| Egg | Milk | ... | Sick? |
|-----|------|-----|-------|
| 0 | 0.7 | | 1 |
| 1 | 0.7 | | 1 |
| 1 | 0.6 | | 1 |
| 2 | 0.6 | | 1 |
| 0 | 1 | | 1 |
| 1 | 0.6 | | 0 |

Fit a decision stump to each leaf's data.
Then add these stumps to the tree.



Greedy Recursive Splitting

This gives a “depth 2” decision tree:



It splits the two datasets into four datasets:

milk ≤ 0.5 data

| Egg | Milk | ... | Sick? |
|-----|------|-----|-------|
| 0 | 0.5 | | 0 |
| 1 | 0 | | 0 |
| 2 | 0 | | 1 |
| 0 | 0.3 | | 0 |
| 2 | 0 | | 1 |

milk > 0.5 data

| Egg | Milk | ... | Sick? |
|-----|------|-----|-------|
| 0 | 0.7 | | 1 |
| 1 | 0.7 | | 1 |
| 1 | 0.6 | | 1 |
| 2 | 0.6 | | 1 |
| 0 | 1 | | 1 |
| 1 | 0.6 | | 0 |

milk ≤ 0.5, egg ≤ 1

| Egg | Milk | ... | Sick? |
|-----|------|-----|-------|
| 0 | 0.5 | | 0 |
| 1 | 0 | | 0 |
| 0 | 0.3 | | 0 |

milk < 0.5, egg > 1

| Egg | Milk | ... | Sick? |
|-----|------|-----|-------|
| 2 | 0 | | 1 |
| 2 | 0 | | 1 |

milk > 0.5, lactase ≤ 0

| Egg | Milk | ... | Sick? |
|-----|------|-----|-------|
| 0 | 0.7 | | 1 |
| 1 | 0.7 | | 1 |
| 1 | 0.6 | | 1 |
| 2 | 0.6 | | 1 |

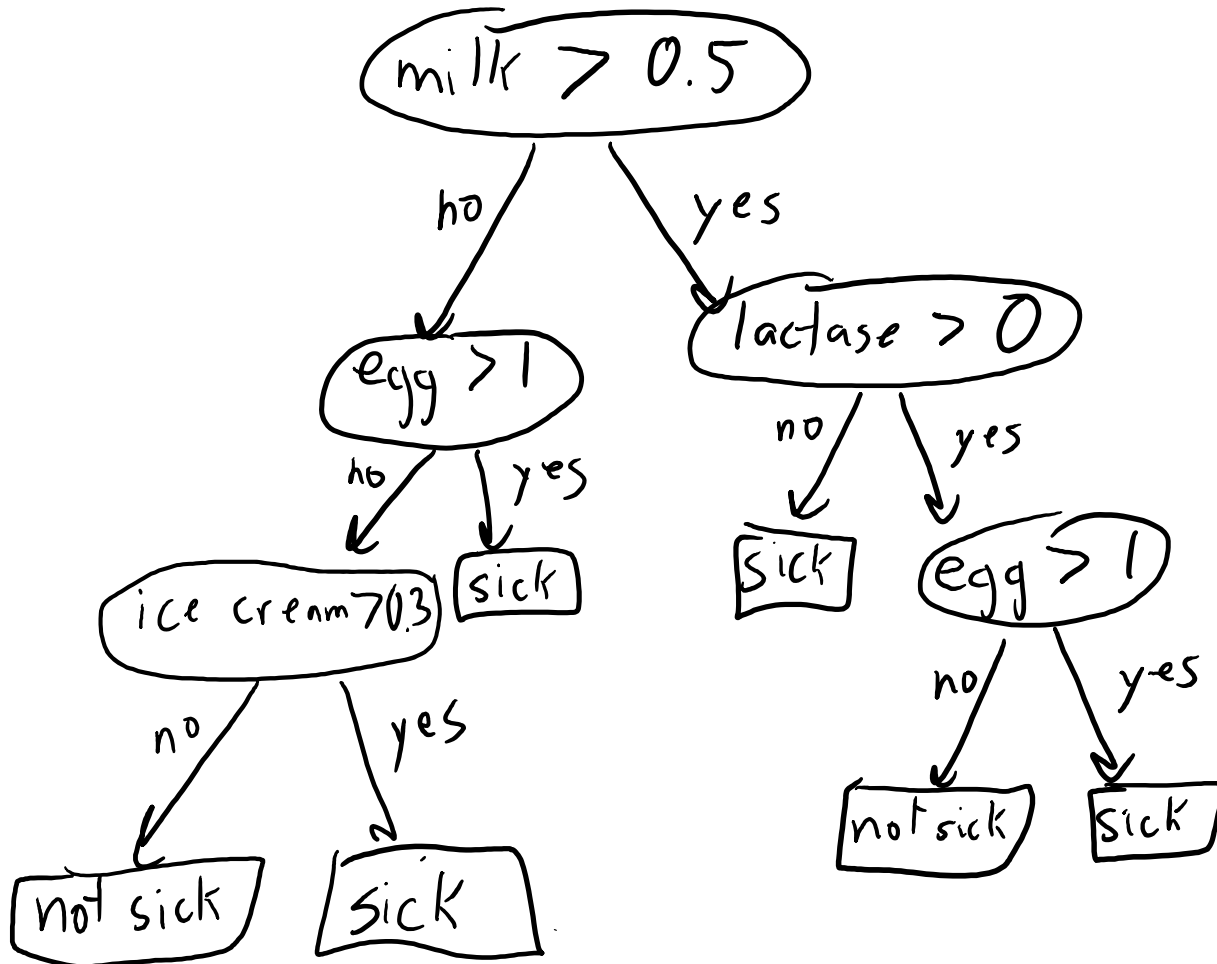
milk > 0.5, lactase > 0

| Egg | Milk | ... | Sick? |
|-----|------|-----|-------|
| 1 | 0.6 | | 0 |

Much more accurate!

Greedy Recursive Splitting

We could try to split the four leaves to make a “depth 3” decision tree:



We might continue splitting until:

- The leaves each have only one label.
- We reach a user-defined maximum depth.

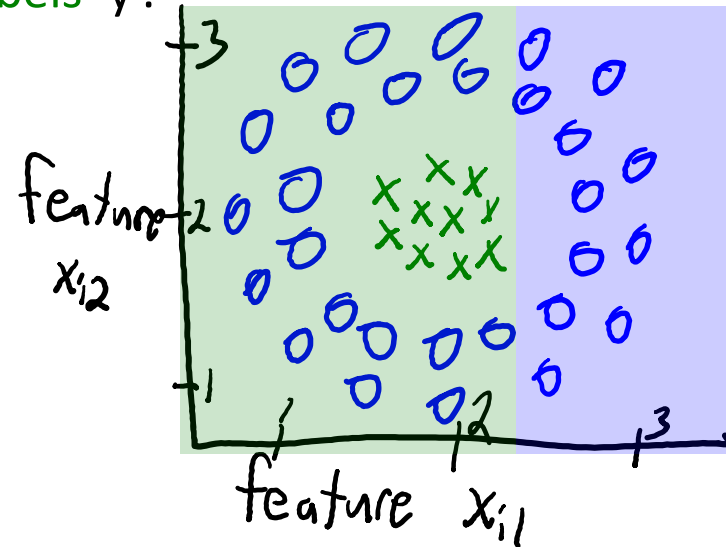
Which score function should a decision tree use?

- Shouldn't we just use accuracy score?
 - For leafs: yes, just maximize accuracy.
 - For internal nodes: not necessarily.
- Maybe no simple rule like $(\text{egg} > 0.5)$ improves accuracy.
 - But this doesn't necessarily mean we should stop!

Example Where Accuracy Fails

- Consider a dataset with 2 features and 2 classes ('x' and 'o').
 - Because there are 2 features, we can draw 'X' as a scatterplot.
 - Colours and shapes denote the class labels 'y'.

$$X = \begin{bmatrix} 1.2 & 2.1 \\ 3.3 & 1.4 \\ 2.0 & 2.1 \\ 2.2 & 2.1 \\ 4.0 & 3.4 \\ \vdots & \vdots \end{bmatrix} \quad Y = \begin{bmatrix} 'o' \\ 'o' \\ 'x' \\ 'x' \\ 'o' \\ \vdots \end{bmatrix}$$



- A decision stump would divide space by a horizontal or vertical line.
 - Testing whether $x_{i1} > t$ or whether $x_{i2} > t$.
- On this dataset no horizontal/vertical line improves accuracy.
 - Baseline is 'o', but need to get many 'o' wrong to get one 'x' right.

Which score function should a decision tree use?

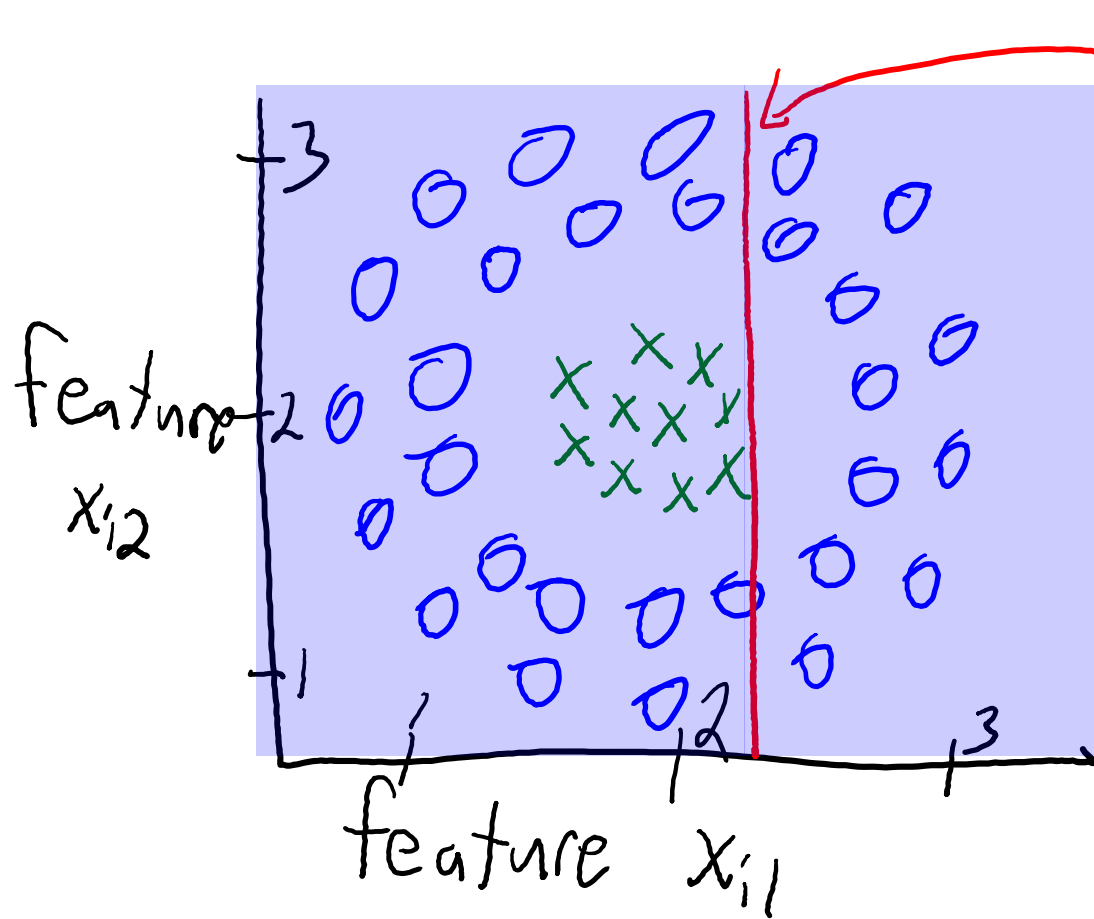
- Most common score in practice is “**information gain**”.
 - “Choose split that decreases **entropy** of labels the most”.

$$\text{information gain} = \underbrace{\text{entropy}(y)}_{\text{entropy of labels before split}} - \frac{n_{\text{leaf1}}}{n} \text{entropy}(y_{\text{leaf1}}) - \frac{n_{\text{leaf2}}}{n} \text{entropy}(y_{\text{leaf2}})$$

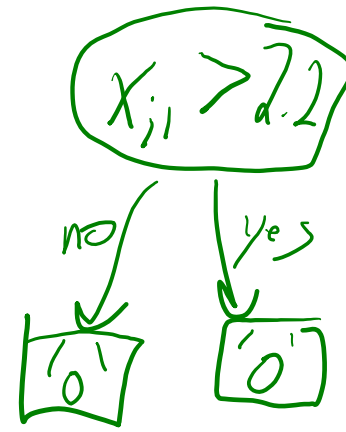
Handwritten annotations:
- An arrow points from the text "number of examples assigned to leaf 1" to the fraction $\frac{n_{\text{leaf1}}}{n}$.
- An arrow points from the text "entropy of labels of examples assigned to leaf 1" to the term $\text{entropy}(y_{\text{leaf1}})$.

- Information gain is always positive, and for baseline rule (“do nothing”) it is 0.
 - Can beat baseline rule whenever at least two non-identical x_i have different labels.
 - Infogain is large if labels are “more predictable” (“less random”) in next layer.
- Even if it does not increase classification accuracy at one depth, we hope that it **makes classification easier at the next depth**.

Example Where Accuracy Fails



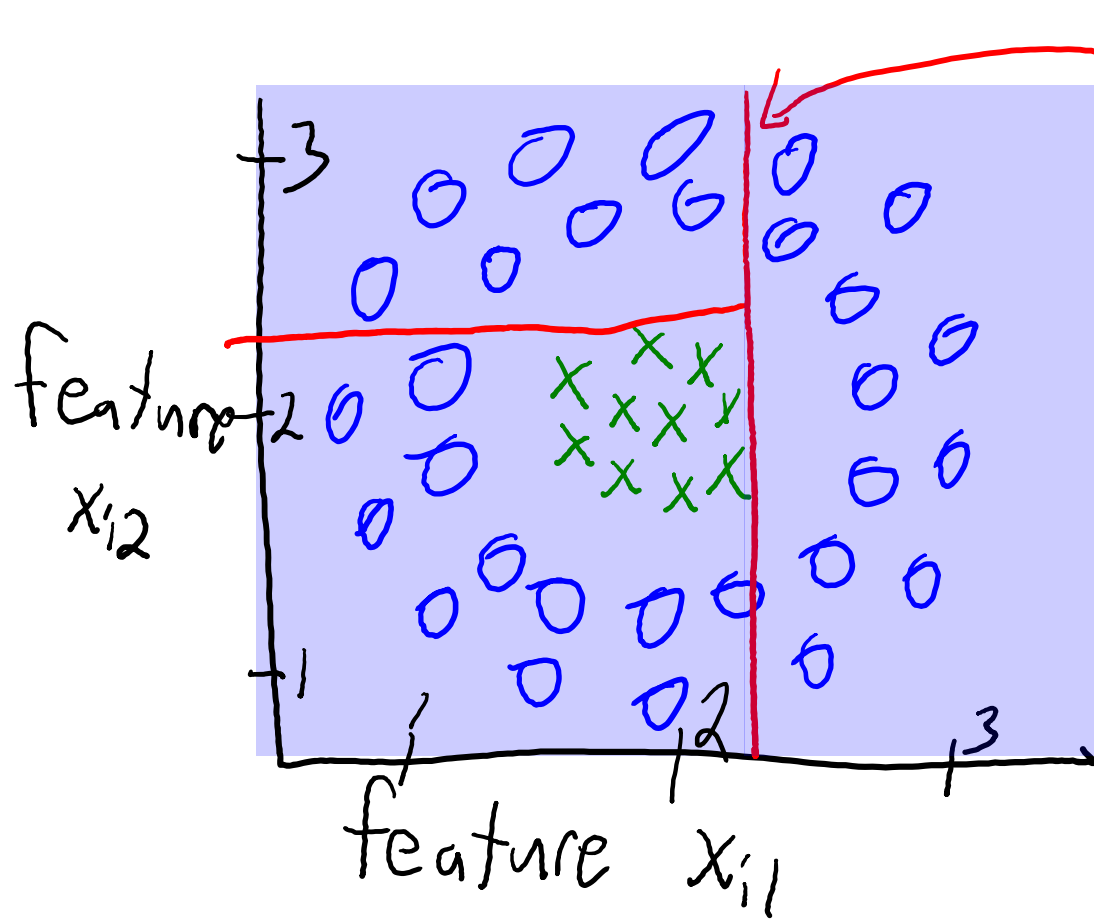
This split makes labels less random.
(Everything on the right is a '0')



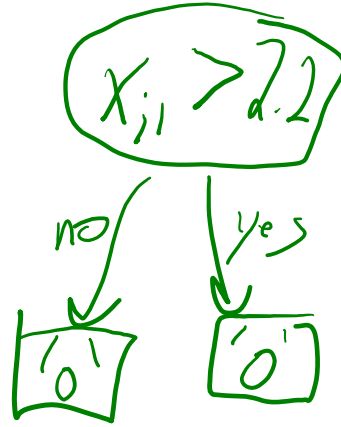
It did not improve accuracy.
(still classifies everything as '0')

But three more splits maximizing info gain lead to perfect accuracy.

Example Where Accuracy Fails



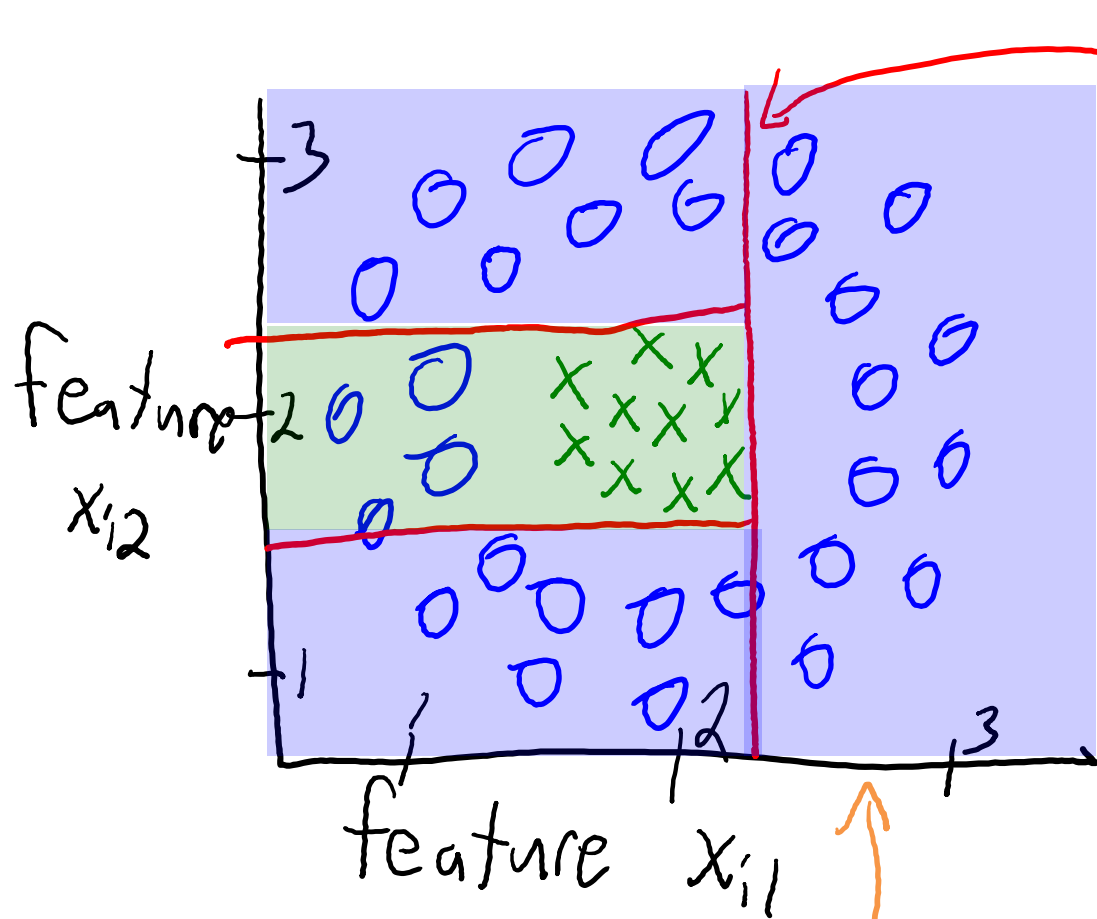
This split makes labels less random.
(Everything on the right is a '0')



It did not improve accuracy.
(still classifies everything as '0')

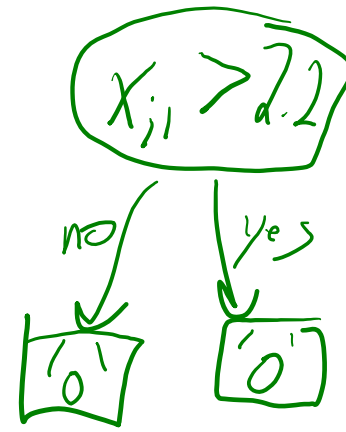
But three more splits maximizing info gain lead to perfect accuracy.

Example Where Accuracy Fails



Using colors to show decisions in regions of space

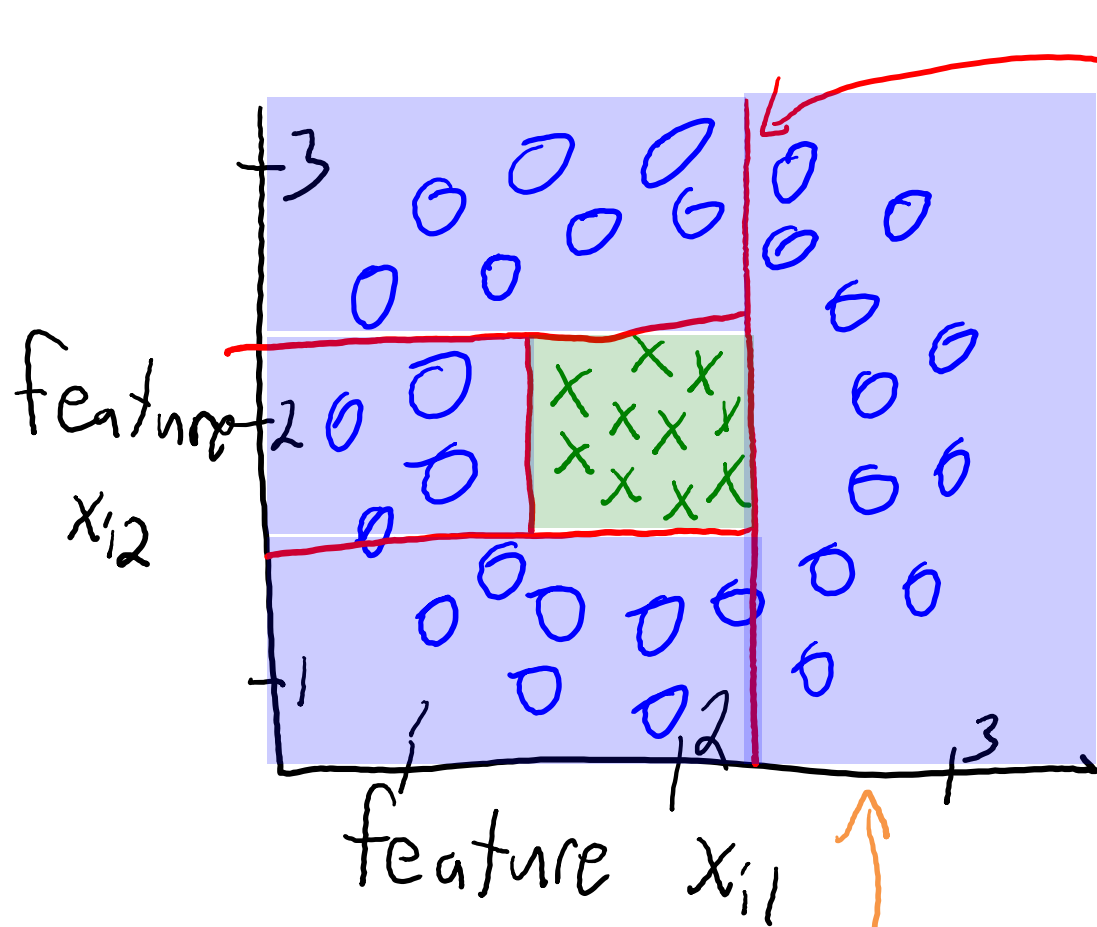
This split makes labels less random.
(Everything on the right is a '0')



It did not improve accuracy.
(still classifies everything as '0')

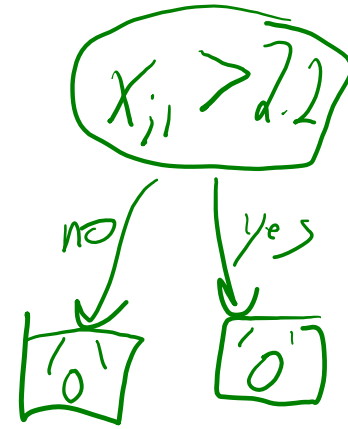
But three more splits maximizing info gain lead to perfect accuracy.

Example Where Accuracy Fails



Using colors to show decisions in regions of space

This split makes labels less random.
(Everything on the right is a '0')



It did not improve accuracy.
(still classifies everything as '0')

But three more splits maximizing info gain lead to perfect accuracy.

Discussion of Decision Tree Learning

- Advantages:
 - Easy to implement.
 - Interpretable.
 - Learning is fast prediction is very fast.
 - Can handle missing values during training (many different things you can do, which you can google if you like)
- Disadvantages:
 - Hard to find optimal set of rules.
 - Greedy splitting often not accurate, requires very deep trees.

Discussion of Decision Tree Learning

- Issues:

- Can you revisit a feature?

- Yes, knowing other information could make feature relevant again
- Or you could be sick if <1 OR >5 eggs (need two splits to do that) [this happened on the x and o example]

- More complicated rules?

- Yes, but searching for the best rule gets much more expensive.

- What is best score?

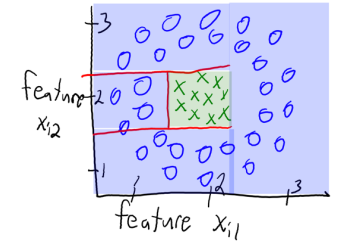
- Infogain is the most popular and often works well, but is not always the best.

- What if you get new data?

- You could consider splitting if there is enough data at the leaves, but occasionally might want to re-learn the whole tree or sub-trees.

- What depth?

- Some implementations stop at a maximum depth.
- Some stop if too few examples in leaf.
- Some stop if infogain is too small.
- Some **stop by checking performance on a “validation set”** (we will discuss this next time).



Summary

- **Supervised learning:**
 - Using data to write a program based on input/output examples.
- **Decision trees:** predicting a label using a sequence of simple rules.
 - Interpretable (at least for individual samples)
- **Decision stumps:** simple decision tree that is very fast to fit.
- **Greedy recursive splitting:** uses a sequence of stumps to fit a tree.
 - Very fast, but not always the most accurate.
- **Information gain:** splitting score based on decreasing entropy.
- Next time: the most important ideas in machine learning.

Entropy Function

Input: vector 'y' of length 'n' with numbers $\{1, 2, \dots, k\}$

```
counts = zeros(k)
```

```
for i in 1:n
```

```
    counts[y[i]] += 1
```

```
entropy = 0
```

```
for c in 1:k
```

```
    prob = counts[c]/n
```

```
    entropy -= prob * log(prob)
```

```
return entropy
```

Other Considerations for Food Allergy Example

- What types of **preprocessing** might we do?
 - **Data cleaning**: check for and fix missing/unreasonable values.
 - **Summary statistics**:
 - Can help identify “unclean” data.
 - Correlation might reveal an obvious dependence (“sick” \Leftrightarrow “peanuts”).
 - **Data transformations**:
 - Convert everything to same scale? (e.g., grams)
 - Add foods from day before? (maybe “sick” depends on multiple days)
 - Add date? (maybe what makes you “sick” changes over time).
 - **Data visualization**: look at a scatterplot of each feature and the label.
 - Maybe the visualization will show something weird in the features.
 - Maybe the pattern is really obvious!
- What you do might depend on how much data you have:
 - Very little data:
 - Represent food by common allergic ingredients (lactose, gluten, etc.)?
 - Lots of data:
 - Use more fine-grained features (bread from bakery vs. hamburger bun)?

Julia Decision Stump Code (not $O(n \log n)$ yet)

Input: feature matrix X and label vector y

$(n, d) = \text{size}(X)$

$\text{minError} = \text{sum}(y \neq \text{mode}(y))$ compute error if you don't split (user-defined function "mode")

$\text{minRule} = []$

for $j = 1:d$

for each feature 'j'

for $i = 1:n$

for each example 'i'

$t = X[i, j]$

set threshold to feature 'j' in example 'i'.

$y_{\text{-above}} = \text{mode}(y[X[:, j]. > t])$

find mode of label vector when feature 'j' is above threshold.

$y_{\text{-below}} = \text{mode}(y[X[:, j]. \leq t])$

find mode of label vector when feature 'j' is below threshold.

$\hat{y} = \text{fill}(y_{\text{-above}}, n)$

Classify all examples based on threshold

$\hat{y}[X[:, j]. \leq t] = y_{\text{-below}}$

$\text{error} = \text{sum}(\hat{y} \neq y)$

count the number of errors.

if $\text{error} < \text{minError}$
 $\text{minError} = \text{error}$
 $\text{minRule} = [j \ t]$

store this rule if it has the lowest error so far.

Going from $O(n^2d)$ to $O(nd \log n)$ for Numerical Features

- Do we have to compute score from scratch?
 - As an example, assume we eat integer number of eggs:
 - So the rules $(\text{egg} > 1)$ and $(\text{egg} > 2)$ have same decisions, except when $(\text{egg} == 2)$.
- We can actually compute the best rule involving 'egg' in $O(n \log n)$:
 - Sort the examples based on 'egg', and use these positions to re-arrange 'y'.
 - Go through the sorted values in order, updating the counts of #sick and #not-sick that both satisfy and don't satisfy the rules.
 - With these counts, it's easy to compute the classification accuracy (see bonus slide).
- Sorting costs $O(n \log n)$ per feature.
- Total cost of updating counts is $O(n)$ per feature.
- Total cost is reduced from $O(n^2d)$ to $O(nd \log n)$.
- This is a good runtime:
 - $O(nd)$ is the size of data, same as runtime up to a log factor.
 - We can apply this algorithm to huge datasets.

How do we fit stumps in $O(nd \log n)$?

- Let's say we're trying to find the best rule involving milk:

| Egg | Milk | ... |
|-----|------|-----|
| 0 | 0.7 | |
| 1 | 0.7 | |
| 0 | 0 | |
| 1 | 0.6 | |
| 1 | 0 | |
| 2 | 0.6 | |
| 0 | 1 | |
| 2 | 0 | |
| 0 | 0.3 | |
| 1 | 0.6 | |
| 2 | 0 | |

| Sick? |
|-------|
| 1 |
| 1 |
| 0 |
| 1 |
| 0 |
| 1 |
| 1 |
| 1 |
| 0 |
| 0 |
| 1 |

First grab the milk column and sort it (using the sort positions to re-arrange the sick column). This step costs $O(n \log n)$ due to sorting.

Now, we'll go through the milk values in order, keeping track of #sick and #not sick that are above/below the current value. E.g., #sick above 0.3 is 5.

With these counts, accuracy score is (sum of most common label above and below)/n.

| Milk |
|------|
| 0 |
| 0 |
| 0 |
| 0 |
| 0.3 |
| 0.6 |
| 0.6 |
| 0.6 |
| 0.6 |
| 0.7 |
| 0.7 |
| 1 |

| Sick? |
|-------|
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 1 |
| 1 |
| 0 |
| 1 |
| 1 |
| 1 |

How do we fit stumps in $O(nd \log n)$?

| Milk | Sick? |
|------|-------|
| 0 | 0 |
| 0 | 0 |
| 0 | 0 |
| 0 | 0 |
| 0.3 | 0 |
| 0.6 | 1 |
| 0.6 | 1 |
| 0.6 | 0 |
| 0.7 | 1 |
| 0.7 | 1 |
| 1 | 1 |

Start with the baseline rule (\emptyset) which is always “satisfied”:

If satisfied, #sick=5 and #not-sick=6.

If not satisfied, #sick=0 and #not-sick=0.

This gives accuracy of $(6+0)/n = 6/11$.

Next try the rule ($\text{milk} > 0$), and update the counts based on these 4 rows:

If satisfied, #sick=5 and #not-sick=2.

If not satisfied, #sick=0 and #not-sick=4.

This gives accuracy of $(5+4)/n = 9/11$, which is better.

Next try the rule ($\text{milk} > 0.3$), and update the counts based on this 1 row:

If satisfied, #sick=5 and #not-sick=1.

If not satisfied, #sick=0 and #not-sick=5.

This gives accuracy of $(5+5)/n = 10/11$, which is better.

(and keep going until you get to the end...)

How do we fit stumps in $O(nd \log n)$?

| Milk | Sick? |
|------|-------|
| 0 | 0 |
| 0 | 0 |
| 0 | 0 |
| 0 | 0 |
| 0.3 | 0 |
| 0.6 | 1 |
| 0.6 | 1 |
| 0.6 | 0 |
| 0.7 | 1 |
| 0.7 | 1 |
| 1 | 1 |

Notice that for each row, updating the counts only costs $O(1)$. Since there are $O(n)$ rows, total cost of updating counts is $O(n)$.

Instead of 2 labels (sick vs. not-sick), consider the case of ' ℓ ' labels:

- Updating the counts still costs $O(n)$, since each row has one label.
- But computing the 'max' across the labels costs $O(\ell)$, so cost is $O(n\ell)$.

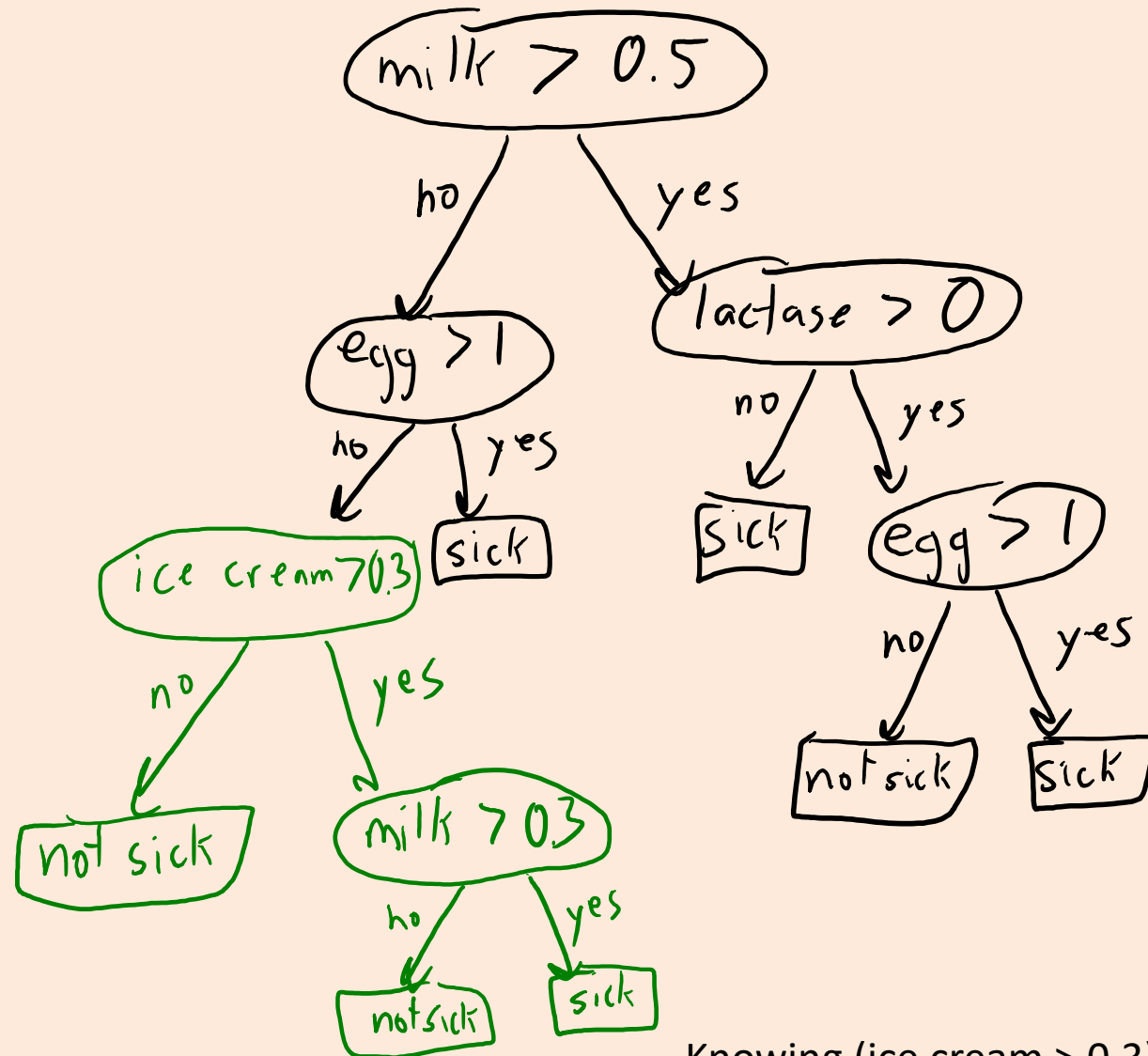
With ' ℓ ' labels, you can decrease cost using a "max-heap" data structure:

- Cost of getting max is $O(1)$, cost of updating heap for a row is $O(\log \ell)$.
- But $\ell \leq n$ (each row has only one label).
- So cost is in $O(\log n)$ for one row.

Since the above shows we can find best rule in one column in $O(n \log n)$, total cost to find best rule across all 'd' columns is $O(nd \log n)$.

Can decision trees re-visit a feature?

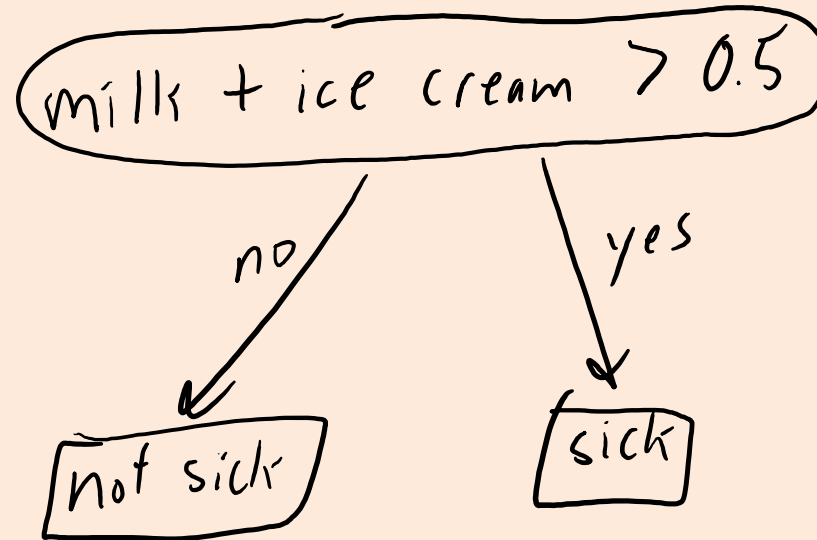
- Yes.



Knowing (ice cream > 0.3) makes small milk quantities relevant.

Can decision trees have more complicated rules?

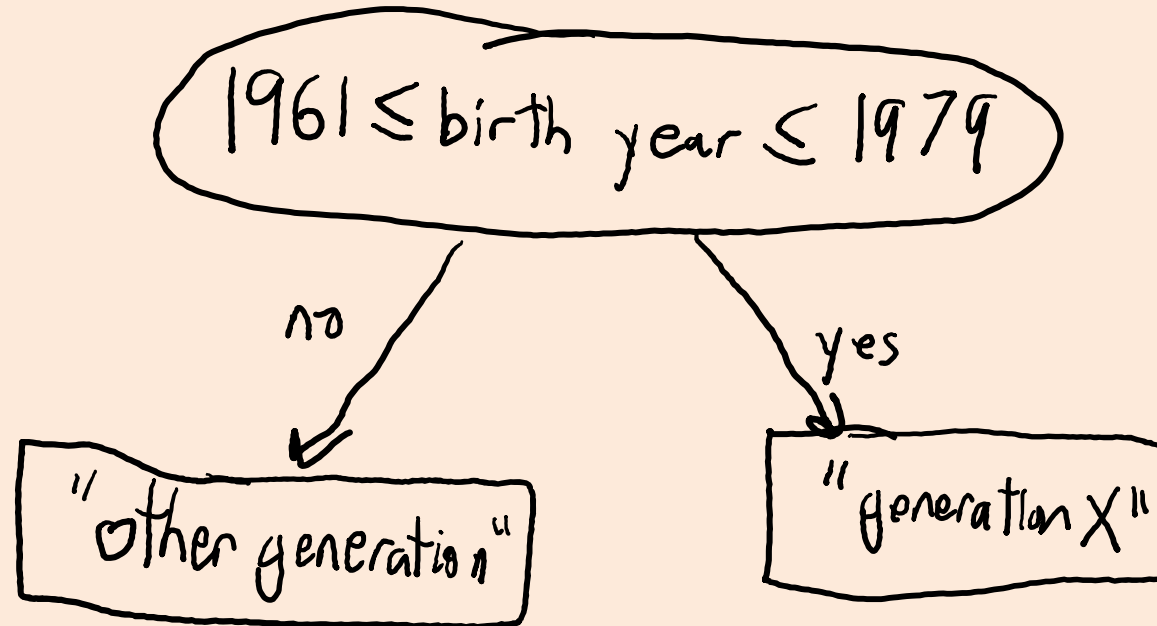
- Yes!
- Rules that depend on **more than one feature**:



- But now searching for the best rule can get expensive.

Can decision trees have more complicated rules?

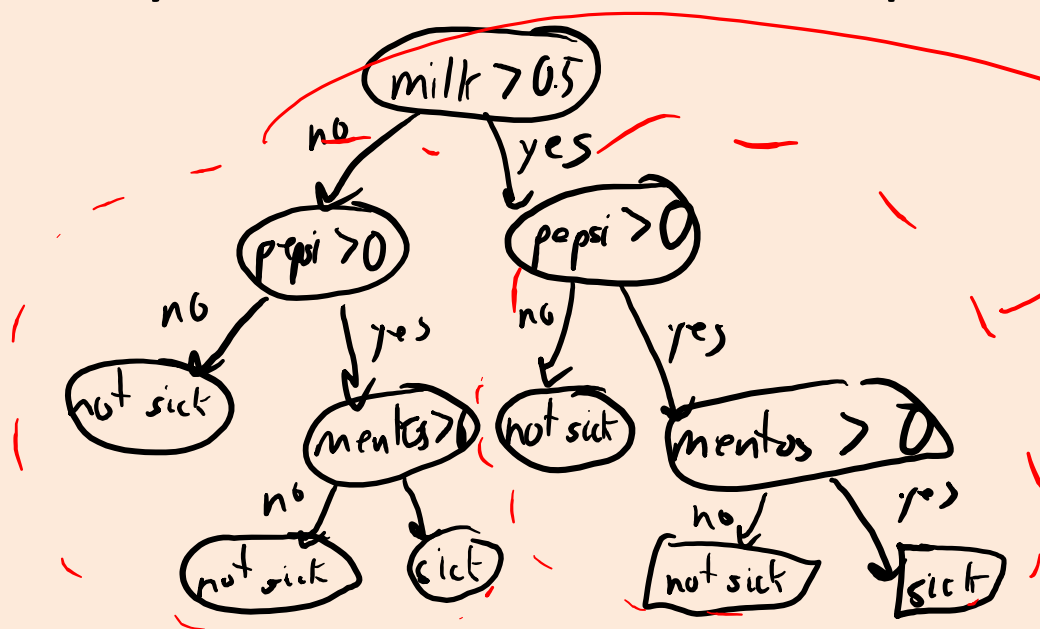
- Yes!
- Rules that depend on **more than one threshold**:



- [“Very Simple Classification Rules Perform Well on Most Commonly Used Datasets”](#)
 - Consider decision stumps based on multiple splits of 1 attribute.
 - Showed that this gives comparable performance to more-fancy methods on many datasets.

Does being greedy actually hurt?

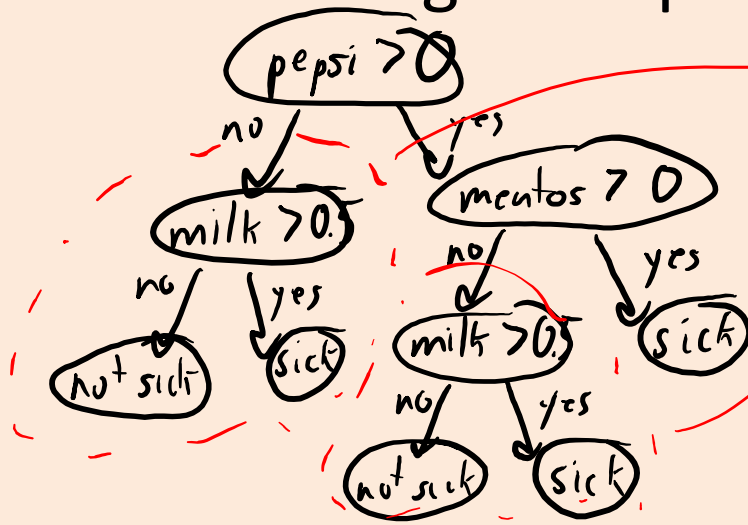
- Can't you just go deeper to correct greedy decisions?
 - Yes, but you need to “re-discover” rules with less data.
- Consider that you are allergic to milk (and drink this often), and also get sick when you (rarely) combine diet coke with mentos.
- Greedy method should first split on milk (helps accuracy the most):



Get same sub-tree,
you need learn it twice and
with less data.

Does being greedy actually hurt?

- Can't you just go deeper to correct greedy decisions?
 - Yes, but you need to “re-discover” rules with less data.
- Consider that you are allergic to milk (and drink this often), and also get sick when you (rarely) combine diet coke with mentos.
- Greedy method should first split on milk (helps accuracy the most).
- Non-greedy method could get simpler tree (split on milk later):



Still has repeated structure, but you should have more data to estimate those splits.

Does being greedy actually hurt?

- Can't you just go deeper to correct greedy decisions?
 - Yes, but you need to “re-discover” rules with less data.
- Consider that you are allergic to milk (and drink this often), and also get sick when you (rarely) combine diet coke with mentos.
- Greedy method should first split on milk (helps accuracy the most).
- Non-greedy method could get simpler tree (split on milk later):
 - [Example](#) of a non-greedy method (complicated and not popular currently).

Decision Trees with Probabilistic Predictions

- Often, we'll have multiple 'y' values at each leaf node.
- In these cases, we might **return probabilities** instead of a label.
- E.g., if in the leaf node we 5 have "sick" examples and 1 "not sick":
 - Return $p(y = \text{"sick"} \mid x_i) = 5/6$ and $p(y = \text{"not sick"} \mid x_i) = 1/6$.
- In general, a natural estimate of the probabilities at the leaf nodes:
 - Let ' n_k ' be the number of examples that arrive to leaf node 'k'.
 - Let ' n_{kc} ' be the number of times ($y == c$) in the examples at leaf node 'k'.
 - Maximum likelihood estimate for this leaf is $p(y = c \mid x_i) = n_{kc}/n_k$.

Alternative Stopping Rules

- There are more complicated rules for deciding when **not** to split.
- Rules based on **minimum sample size**.
 - Don't split any nodes where the number of examples is less than some 'm'.
 - Don't split any nodes that create children with less than 'm' examples.
 - These types of rules try to make sure that you have enough data to justify decisions.
- Alternately, you can use a **validation set** (see next lecture):
 - Don't split the node if it decreases an approximation of test accuracy.