Need $\geq n!$ leaves

Else, cannot distinguish all outputs.

A binary tree of depth d
has $\leq 2^d$ leaves.

Our tree must have depth
sufficiently large that

$$\lg 2^d \geq \lg n!$$

$$d \geq \lg(n!)$$

So, SORTING BY COMPARISON
is IN $\Omega(\lg(n!))$

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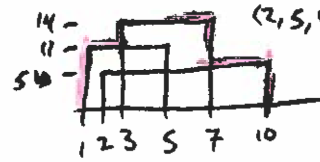
SKYLINE PROBLEM

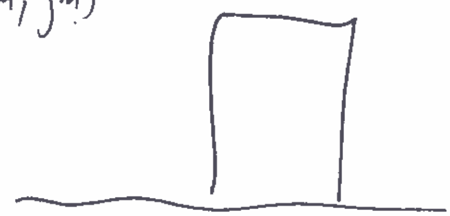
INPUT: List of bldgs $B[1 \ldots n]$
triples (left, height, right)

EXAMPLE: (1,11,5), (3,14,7), (2,5,10)



OUTPUT: (1,11), (3,14), (7,5), (10,

OUTPUT: SHAPE OF SKYLINE
$(x_1,y_1),(x_2,y_2),\ldots,(x_m,y_m)$ 0



DIVIDE-AND-CONQUER

$T(n)$: time for SKY on n bldgs

SKYLINE-ALG(B)

(1) IF THERE'S ONE BLDG (x_1,h,x_2)
OUTPUT $(x_1,h),(x_2,0)$

$T(1)=1$

for $n>1$ $T(n) = T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + cn$

(2) ELSE, DIVIDE INTO

$n - \lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil$

$\lfloor \frac{n}{2} \rfloor$ $B[1\ldots \lfloor \frac{n}{2} \rfloor] + B[\lfloor \frac{n}{2} \rfloor+1 \ldots n]$ floors/ceilings?

(a) OUT_1 = SKYLINE_ALG$(B[1\ldots \lfloor \frac{n}{2} \rfloor])$ — $T(\lfloor \frac{n}{2} \rfloor)$
(b) OUT_2 = SKYLINE_ALG$(B[\lfloor \frac{n}{2} \rfloor+1 \ldots n])$ — $T(\lceil \frac{n}{2} \rceil)$
(c) MERGE OUT_1 and OUT_2 into OUTPUT

$c \cdot n$
c just for fun

$O(|B|)$

MERGE(OUT_1, OUT_2)
$h_1=0$, $h_2=0$, $curr_x=0$, $curr_y=0$
WHILE OUT_1 OR OUT_2 is not EMPTY DO
IF OUT_1 AND OUT_2 START AT THE SAME X-COORD
$curr_x$ = FIRST X-COORD IN OUT_1
h_1 = FIRST Y-COORD IN OUT_1
h_2 = FIRST Y-COORD IN OUT_2
DELETE FIRST ENTRIES IN OUT_1/OUT_2
ELSE IF OUT_1 HAS SMALLEST FIRST X-COORD
$curr_x$ = FIRST X-COORD IN OUT_1
h_1 = FIRST Y-COORD IN OUT_1
(2) DELETE FIRST ENTRY IN OUT_1
ELSE
$curr_x$ = FIRST X-COORD I
h_2 = FIRST Y-COORD IN
DELETE FIRST ENTRY I
IF $\max(h_1,h_2) \neq curr_y$
APPEND $(curr_x, \max(h_1,$
TO OUTPUT
$curr_y = \max(h_1,h_2)$
END
APPEND $(curr_x,0)$ TO OUTP

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$$T(n) = 2\underbrace{T\left(\frac{n}{2}\right)} + cn \qquad T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + c\frac{n}{2}$$

$$= 2\left(2T\left(\frac{n}{4}\right) + c\frac{n}{2}\right) + cn \qquad T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + c\frac{n}{4}$$

$$= 4\underbrace{T\left(\frac{n}{4}\right)} + cn + cn$$

$$= 4\left(2T\left(\frac{n}{8}\right) + c\frac{n}{4}\right) + cn + cn$$

$$= 8T\left(\frac{n}{8}\right) + cn + cn + cn$$

$$= 2^k T\left(\frac{n}{2^k}\right) + kcn \quad \text{for } k-1 \text{ substitutions}$$

~~Let~~ Want $\frac{n}{2^k} = 1$

$\lg n = \lg 2^k$

$\lg n = k$

$$= nT(1) + (\lg n)cn$$

$$= n + cn\lg n \qquad \text{NOT YET PROVEN}$$

Prove $T(n) = n + cn\lg n$ for n a power of 2.

By induction over powers of 2

BC: $T(1) = 1 + c \cdot 1 \cdot \lg 1 = 1 + 0 = 1$ ✓

IH: $\boxed{T(2^k)}$ for $k \geq 0$ $= 2^k + c2^k \lg 2^k = 2^k + ck2^k$

IS: Show $T(2^{k+1}) = 2^{k+1} + c2^{k+1}\lg 2^{k+1} = 2^{k+1} + c(k+1)2^{k+1}$

$$T(2^{k+1}) = 2T\left(\frac{2^{k+1}}{2}\right) + c2^{k+1} = 2T(2^k) + c2^{k+1}$$

$$= 2(2^k + ck2^k) + c2^{k+1} = 2^{k+1} + ck2^{k+1} + c2^{k+1}$$

$$= 2^{k+1} + c2^{k+1}(k+1) \checkmark$$

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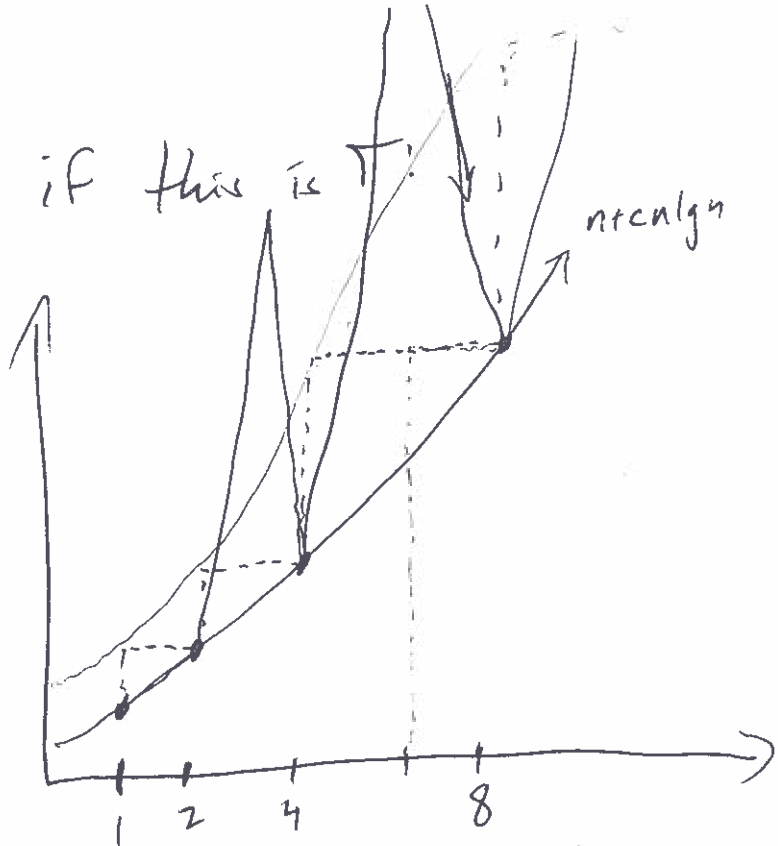
What if this is T:



Prove by induction that $T(n) \le T(n+1)$ for all $n \in \mathbb{N}$

BC: Show $T(1) \le T(2)$

$T(1) = 1$

$T(2) = T(1) + T(1) + c2$

$= 1 + 1 + 2c = 2c + 2$

~~And $1 \le 4$.~~

We know $c > 0$; so

$2c + 2 \ge 2 \ge 1$ ✓.

IH: $T(k) \le T(k+1)$ for all $k < n$

IS: Show $T(n) \le T(n+1)$

Know $n > 1$

$T(n) = \boxed{T\left(\left\lceil \frac{n}{2} \right\rceil\right)} + \boxed{T\left(\left\lfloor \frac{n}{2} \right\rfloor\right)} + \boxed{cn}$

by IH $\le$ $\quad\le\quad\le$

$T(n+1) = \boxed{T\left(\left\lceil \frac{n+1}{2} \right\rceil\right)} + \boxed{T\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right)} + \boxed{c(n+1)}$

either = or 1 apart

$T(n) \le T(n+1)$.

QED