CPSC 320
2010/01/12

$$O(f(n)) = \{g : \exists c \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \ge n_0 \to g(n) \le c f(n)\}$$

Prove: $13n^2 + 3n \in O(n^2)$

Pick a c: 16

Pick an n_0: 1

Show $13n^2 + 3n \le cn^2$ for all $n \ge n_0$

$$13n^2 + 3n \le \cancel{16n}\ 13n^2 + 3n^2 \quad \text{since } n \ge 1$$
$$= 16n^2$$
$$= cn^2 \ ✓$$

QED

SCRATCH: ($n \ge 1$)

$$13n^2 + 3n \le 13n^2 + 3n^2 \quad n_0$$
$$= \underbrace{16}_{c} n^2$$

Prove: $13n^2 + 3n \notin O(n)$

Assume for contradiction:

$13n^2 + 3n \in O(n)$

$\exists c \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \ge n_0 \to 13n^2 + 3n \le cn$

Let $n > \max\left(n_0, \frac{c-3}{13}\right)$

Then, $\boxed{n > \frac{c-3}{13}}$ and $n \ge n_0$

So, $\boxed{13n^2 + 3n \le cn}$

But, $13n^2 + 3n > 13n\left(\frac{c-3}{13}\right) + 3n$ So, $13n^2 + 3n > cn$.

$$= n(c-3) + 3n$$
$$= cn - 3n + 3n$$
$$= cn$$

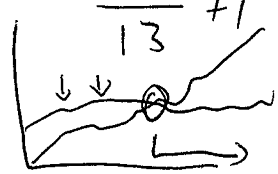
CONTRADICTION

QED.

1

SCRATCH:

$$13n^2 + 3n = cn$$
$$13n + 3 = c$$
$$13n = c - 3$$
$$n = \frac{c-3}{13}$$
$$n > \left\lceil \frac{c-3}{13} \right\rceil$$
$$n = \left\lceil \frac{c-3}{13} \right\rceil + 1$$



Prove $3^n \notin O(2^n)$

Assume for contradiction that

$$3^n \in O(2^n)$$

Then, $\exists c, n_0, \forall n \geq n_0, 3^n \leq c2^n$

Let $n > \max(n_0, \log_{3/2} c)$

So, $3^n \leq c2^n \Rightarrow c \geq \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n$

$\cancel{3^n > 3^{\log_{3/2} c}}$

$$c \geq \left(\frac{3}{2}\right)^n > \frac{3}{2}^{\log_{3/2} c} = c$$

So, $c > c$.

CONTRADICTION

QED

SCRATCH:

$3^n = c2^n$

$\boxed{c = \frac{3^n}{2^n}}$ $\Leftarrow$

$c = \left(\frac{3}{2}\right)^n$

$\log_{3/2} c = n$

Prove: If $f(n) \in O(g(n))$ then $d f(n) \in O(g(n))$, where $d \in \mathbb{R}^+$.

Assume $f(n) \in O(g(n))$.

$\exists c, n_0, \forall n \geq n_0, f(n) \leq cg(n)$.

$\forall n \geq n_0', \; d f(n) \leq d \cdot (cg(n))$

$= (dc) g(n)$

$= c' g(n)$.

Pick $c' = dc$

Pick $n_0' = n_0$

QED.

$f(n) \in O(g(n)) \iff g(n) \in \Omega(f(n))$

If $\underbrace{f(n) \in O(h(n))}^{1}$ and $\underbrace{g(n) \in o(f(n))}^{2}$,
then $\underbrace{f(n)+g(n) \in O(h(n))}_{3}$.

KNOW:
$\exists c_1, n_1, \forall n \geq n_1, f(n) \leq c_1 h(n)$.
$\forall c_2, \exists n_2, \forall n \geq n_2, g(n) \leq c_2 f(n)$.

SHOW: $\exists c_3, n_3, \forall n \geq n_3, f(n)+g(n) \leq c_3 h(n)$.

Let $c_2 = 1$. Then, $\exists n_2, \forall n \geq n_2, g(n) \leq f(n)$.

Pick $n_3 = \max(n_2, n_1)$
Pick $c_3 = 2c_1$

Then, $f(n)+g(n) \leq f(n)+f(n)$ since $n \geq n_3 \geq n_2$
$= 2f(n)$
$\leq 2c_1 h(n)$ since $n \geq n_3 \geq n_1$
$= c_3 h(n)$. ✓

QED

SCRATCH:
$f(n)+g(n)$
$g(n) \in o(f(n))$
$f(n)+g(n) \leq f(n)+f(n)$
$= 2f(n)$
$\leq 2c_1 h(n)$

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$\lg n \equiv \log_2 n$

Prove: $2^{2^n} \in \omega(n^n)$

$$\lim_{n\to\infty} \frac{2^{2^n}}{n^n} \Longleftarrow n = 2^?$$

$$n = \cancel{\lg}\, 2^{\lg n}$$

$$= \lim_{n\to\infty} \frac{2^{2^n}}{(2^{\lg n})^n}$$

$$= \lim_{n\to\infty} \frac{2^{2^n}}{2^{n\lg n}}$$

$$= \lim_{n\to\infty} 2^{2^n - n\lg n} \Rightarrow \infty$$

So, $2^{2^n} \in \omega(n^n)$

Prove: $\lg n \in o(\sqrt{n})$

$$\lg n = \frac{\ln n}{\ln 2} \qquad \frac{d\,\frac{\ln n}{\ln 2}}{dn} = \frac{1}{n\ln 2}$$

$$\lim_{n\to\infty} \frac{\lg n}{\sqrt{n}} \to \frac{\infty}{\infty}$$

$$\sqrt{n} = n^{0.5}$$

$$= \lim_{n\to\infty} \frac{\frac{d\lg n}{dn}}{\frac{d\sqrt{n}}{dn}}$$

$$\frac{d\sqrt{n}}{dn} = 0.5 n^{-0.5}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n\ln 2}}{0.5 n^{-0.5}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n\ln 2} = \lim_{n\to\infty} \frac{2}{\sqrt{n}\ln 2} = 0$$

QED