Approximation Algorithms

Final Exam
Dec 11 8:30a
LSK 200

A is a *$\rho(n)$-approximation algorithm* if for every input I of size n with optimal solution value $OPT(I)$:

$$\max\left(\underbrace{\frac{A(I)}{OPT(I)}}_{\text{handles minimizing value}}, \underbrace{\frac{OPT(I)}{A(I)}}_{\text{handles maximizing value}}\right) \le \rho(n)$$

Example (Minimum) Vertex Cover

Find smallest set of vertices $S \subseteq V$ such that all edges in G have at least one endpoint in S.

Approximation algorithm for Vertex Cover

Matching VC

Example

$S = \emptyset$
Pick edge $(u,v) \in G$ (any edge)
Remove u and v and all adjacent edges from G
Add u and v to S
Repeat until G contains no edges
Return S

Claim Matching VC is a 2-approximation alg. for min. VC

proof Let $VC(G)$ be the smallest vertex cover in G.

1. $|VC(G)| \ge$ # edges picked by MVC. Why? Each picked edge must be covered by $VC(G)$ and no two picked edges share the same endpoint.

2. $|MVC(G)| = 2 \times$ #edges picked by MVC

$$\Rightarrow \frac{|MVC(G)|}{|VC(G)|} \le 2$$

Key idea: We don't know how big $VC(G)$ is but we can lower bound its size.

Approximation Algorithm for ΔTSP

The traveling-salesman problem is: Given a complete undirected graph $G=(V,E)$ with a non-negative weight $w(u,v)$ on every edge. Find a simple cycle with $n=|V|$ edges (a tour) of minimum total weight

$E = V \times V$

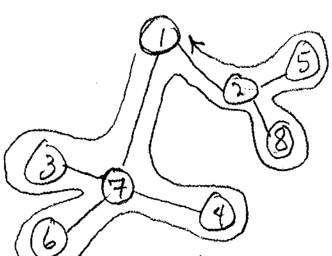
ΔTSP is the same problem but the edge weights obey the triangle inequality:

$$w(a,c) \leq w(a,b) + w(b,c) \quad \text{for all } a,b,c \in V$$

ΔTSP is NP-complete (so is general TSP)

TreeTour (G)

1. Find $T = MST(G)$ (minimum spanning tree)
2. Let L be the list of vertices visited by a pre order traversal of T (using any vertex r as the root)
3. return $L \circ r$ as the tour



$L = 1,7,3,6,4,2,8,5$

$L\circ r = 1,7,3,6,4,2,8,5,1$

Claim TreeTour (G) is a 2-approx. for ΔTSP.

proof Let OPT(G) be the minimum weight tour of G

1. $w(OPT(G)) \geq w(MST(G))$

   If we delete any edge of OPT(G) we get a spanning tree. The min weight spanning tree weighs even less (or the same).

2. Let W be the list of vertices visited and returned to in a preorder traversal. $W = 1,7,3,7,6,7,4,7,1,2,8,2,5,2,1$

$w(W) = 2\,w(MST(G))$ because each edge is traversed twice

$w(L\circ r) < w(W)$ by triangle inequality.

$$\boxed{\text{So } w(\text{TreeTour}(G)) \leq 2\,w(OPT(G)).}$$