Thm CLIQUE is NP-hard

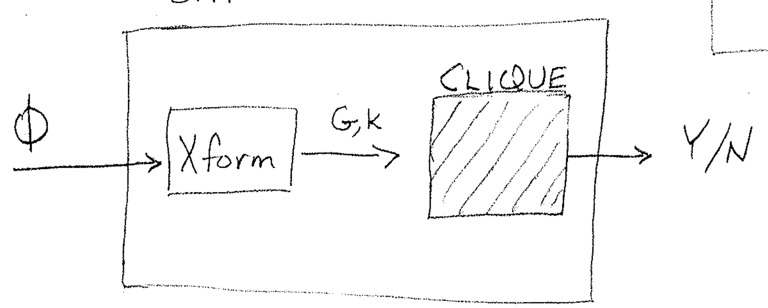
proof We want to show: if CLIQUE $\in P$ then $NP = P$

We know: if SAT $\in P$ then $NP = P$
(from Cook's Theorem)

So, we need only show $\boxed{\text{If CLIQUE} \in P \text{ then SAT} \in P}$

In other words, we want to show how to use a fast algorithm for CLIQUE to build a fast algorithm for SAT.

Reduction from SAT to CLIQUE



We want to transform a boolean formula ϕ into a graph G and integer k so that

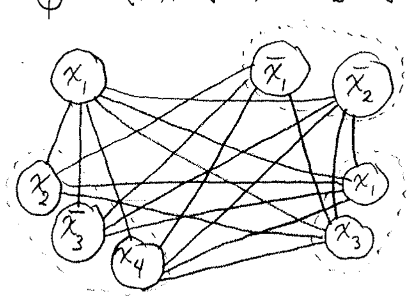
1. G contains a clique of size k iff ϕ is satisfiable
2. the transformation takes polynomial time.

Transform

A. Create a vertex for every literal in every clause
B. Connect every vertex from clause i to every vertex from clause j for all $i \neq j$ unless they are the negation of each other
C. Let k = # clauses in ϕ

Example $\phi = (x_1)(\bar{x}_1 \vee \bar{x}_2)(x_1 \vee x_3)(x_2 \vee \bar{x}_3 \vee x_4)$

G =



Claim $\phi \in SAT$ iff G has a k-clique.

proof

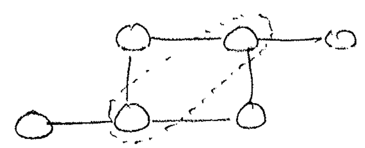
$\Rightarrow$ If $\phi \in SAT$ then there is a truth assignment such that every clause has ≥ 1 true literal. Choose one true literal vertex from each clause and put these vertices in a set Q. $|Q| = k$ and Q is a clique because every pair of vertices in Q come from different clauses and they cannot be the negation of each other (or they wouldn't both be true literals). So there is an edge between them.

$\Leftarrow$ If G has a k-clique Q then exactly one vertex from each clause is in Q. Assign True to every literal with a vertex in Q (and False otherwise). So every clause contains a true literal. and no variable is assigned both True and False (since no literal and its negation can be in Q).

We now know: CLIQUE is NP-complete.

Vertex Cover

A vertex cover is a set of vertices $S \subseteq V$ such that all edges in G have at least one endpoint in S. [i.e. S "covers" the edges]



has a vertex cover of size 2.

What's the smallest Vertex Cover in G?

Related decision problem:

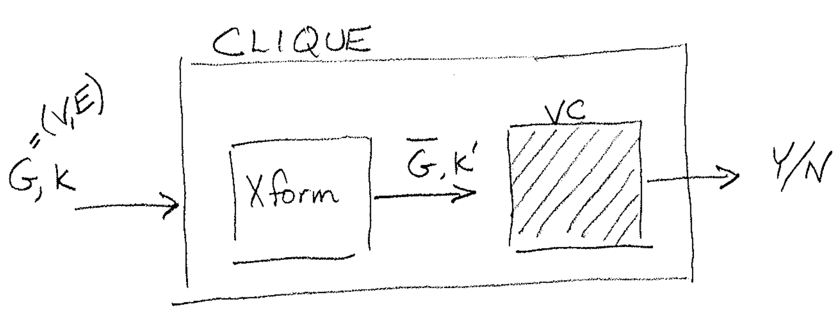
Given graph G and integer k,
Does G have a vertex cover of size k?

$VC = \{ \langle G, k \rangle \mid G \text{ has a vertex cover of size } k \}$

Thm VC is NP-complete

proof ① $VC \in NP$ — A witness is a vertex cover S of size k. The verifier checks that $|S| = k$ and, for each $(u,v) \in E$, that $u \in S$ or $v \in S$. Verification takes polynomial time.

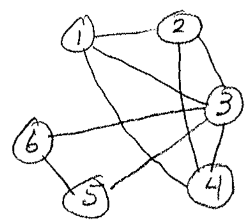
② CLIQUE is reducible (by polytime reduction) to VC.

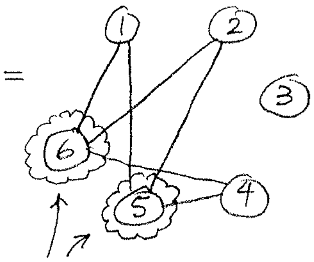


$\overline{G}$ is the complement of G: $\overline{G} = (V, \overline{E})$ where $\overline{E} = \{(u,v) \mid (u,v) \notin E\}$

$k' = |V| - k$

Example

$G =$ 

$\overline{G} =$ 

vertex cover

Claim G has a clique of size k iff $\overline{G}$ has a vertex cover of size $|V|-k$.

proof

$\Rightarrow$ Let $S \subseteq V$ be a clique (of size k) in G.
We show $V-S$ is a vertex cover of $\overline{G}$.
Consider $(u,v) \in \overline{E}$. Since $(u,v) \notin E$ either u or v is not in S $\Rightarrow$ u or v is in $V-S$
$\Rightarrow$ (u,v) is covered by $V-S$.

$\Leftarrow$ Let $R \subseteq V$ be a vertex cover in $\overline{G}$.
We show $V-R$ is a clique in G.
If $(u,v) \in \overline{E}$ then $u \in R$ or $v \in R$ (or both)
(contrapositive)
$\hookrightarrow$ If $u \notin R$ and $v \notin R$ then $(u,v) \notin \overline{E}$
$\hookrightarrow$ If $u,v \in V-R$ then $(u,v) \in E$ $\Rightarrow$ $V-R$ is a clique.