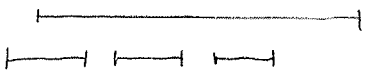
What if we want to maximize <u>total length</u> of the set of nonoverlapping jobs? (job length = $f_i - s_i$)

Previous greedy algorithm doesn't work

Example: 

Given $S = (s_1, f_1)\ (s_2, f_2)\ \ldots\ (s_n, f_n)$

Find set of non-overlapping jobs of maximum total length

<u>Observation</u> An optimal solution for jobs S either

① includes job n and contains an optimal solution for S' = jobs not overlapping job n

or ② doesn't include job n and is an optimal solution for jobs $1 \ldots n-1$

In both cases we have created a subproblem of our original problem, which we can solve by creating more subproblems..

How many subproblems do we need to solve?

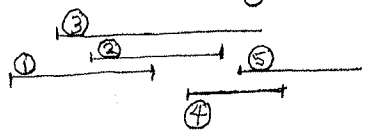
Because of ② we need to solve all problems of the form: Find an optimal set from jobs $1 \ldots i$

What about ①? Can we order the jobs so that S' = jobs $1 \ldots i$ for some i?

Yes! If $f_1 \leq f_2 \leq f_3 \ldots \leq f_n$ then the jobs that don't overlap job n are jobs $1 \ldots i$ for some i.

Let last[j] = largest index less than j of a job that doesn't overlap job j (or 0 if no such job exists)

For example:



last[1] = 0
last[2] = 0
last[3] = 0
last[4] = 1
last[5] = 2

Let L[j] = total length of optimal solution for jobs 1 ... j

Observation $L[n] = \max \{ L[\text{last}[n]] + (f_n - s_n) \text{ (1)}, \; L[n-1] \text{ (2)} \}$

In general: $L[j] = \max \{ L[\text{last}[j]] + (f_j - s_j), \; L[j-1] \}$ for $j = 1, 2, \ldots n$

where $L[0] = 0$

Max Length Sched (S)

Sort jobs so that $f_1 \le f_2 \le \ldots \le f_n$ — O(n lg n)

Calculate last[j] for $j = 1 \ldots n$ — O(n lg n) (How?) Use binary search to find s_j in sorted list of $f_1 f_2 \ldots f_n$

$L[0] = 0$

For $j = 1$ to n — O(n)
  $L[j] = \max \{ L[\text{last}[j]] + (f_j - s_j), L[j-1] \}$

Return L[n]

This returns the value (total length) of the optimal solution.
What if we want the set of jobs?

$W = \emptyset$
$j = n$
while $j > 0$
  if $L[j] > L[j-1]$ then { $W = W \cup \{j\}$; $j = \text{last}[j]$ }
  else { $j = j - 1$ }
Return W

This is an example of Dynamic Programming:

To solve a problem

optimal substructure:
- Find a relation that shows how to solve the problem given solutions to "smaller" subproblems
  (make sure the total number of subproblems is small)
- Solve the subproblems from smaller to larger

Longest Increasing Subsequence

Given a sequence of numbers $A = a_1, a_2 \ldots a_n$

a <u>subsequence</u> of A is a sequence $B = b_1, b_2 \ldots b_k$

such that $b_1 = a_{i_1}$ $b_2 = a_{i_2}$ $\ldots$ $b_k = a_{i_k}$

and $1 \le i_1 < i_2 < \ldots < i_k \le n$

For example: $A = 3\ 5\ 2\ 1\ 7\ 10\ 9$

has subseq. $B = 5\ 1\ 9$ $(i_1 = 2\ \ i_2 = 4\ \ i_3 = 7)$

A sequence $B = b_1, b_2 \ldots b_k$ is <u>increasing</u> if $b_1 < b_2 < \ldots < b_k$

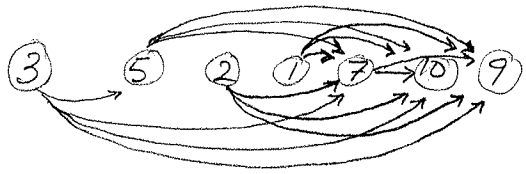
<u>Find</u> a longest increasing subsequence of $A = a_1, a_2 \ldots a_n$.

for $3\ 5\ 2\ 1\ 7\ 10\ 9$ one answer is 3, 5, 7, 9

How do we find LIS(A)?

Consider all possible increasing subsequences?

No. Too many.



Intuition { Find the longest path in this graph
where vertex i has label a_i $\quad i = 1 \ldots n$
and edge (i,j) exists iff $a_i < a_j$ and $i < j$

Let $L[j]$ = length of the longest increasing subsequence of A that ends with a_j

$$L[j] = 1 + \max \{ L[i] \mid i<j \text{ and } a_i < a_j \}$$

Algorithm (to find length of LIS(A))

$O(n^2)$ [for $j = 1$ to n

$O(n) \longrightarrow L[j] = 1 + \max \{ L[i] \mid i<j \text{ and } a_i < a_j \}$

$O(n) \longrightarrow$ return $\max_j L[j]$

An LIS that ends at a_j is an LIS that ends at a_i (for $i<j$ and $a_i < a_j$) followed by a_j.

To find the actual sequence:

- Let k be such that $L[k] = \max_j L[j]$

```
print k
for i = k-1 to 1
    [if a_i < a_k and L[i] = L[k] - 1 then
        [print i
        [k = i
```