Finding k^{th} smallest element

Section 9.2 CLRS

Input An array of n real numbers $A[1 \dots n]$ and $k \in \mathbb{N}$, $k \leq n$

Output k^{th} smallest of A

(Assume distinct. We could break ties by saying $A[i] < A[j]$ if $A[i] = A[j]$ and $i < j$)

Solution 1 (a reduction)

Use sorting to solve the problem.

$A = \text{Sort}(A)$
return $A[k]$

+ foolproof $O(n \log n)$
- $\Omega(n \log n)$

Solution 2

Build a heap H from A ← $O(n)$
Do k times : $x = \text{extractMin}(H)$ ← $O(k \cdot \log n)$
return x

+ foolproof $O(n + k \log n)$
- $\Omega(n + k \log n)$ can be as bad as $\Omega(n \log n)$

Solution 3 (recursion)

Idea: (Quicksortish). split A into small (S) and large (L) elements. Compare $|S|$ to k to choose whether to look in S or L.

RSelect (A, k)
If $|A| = 1$ return $A[1]$
Pick pivot p from $A[1 \dots n]$ at random (each element equally likely)
$S = \{ A[i] \mid A[i] < p \}$
$L = \{ A[i] \mid A[i] > p \}$
If $|S| \geq k$ return RSelect (S, k)
else if $|S| = k-1$ return p
else return RSelect $(L, k - |S| - 1)$

If we could guarantee that the pivot is "in the middle" then the recursive subproblem will be small

If pivot has $\leq (1-\varepsilon)n$ elements smaller than it
and $\leq (1-\varepsilon)n$ elements bigger than it
then the size of the problem decreases by a factor $\leq (1-\varepsilon)$.

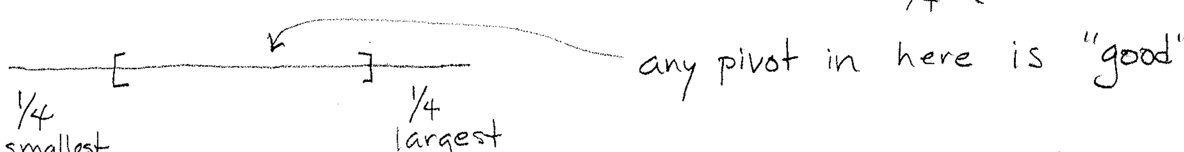
In that case the running time $T(n)$ would be

Assume the worst value of k so that we recurse in the larger of the two possible subproblems.

$$T(n) \leq c \cdot n + T((1-\varepsilon)n)$$
$$\leq c \cdot n + c(1-\varepsilon)n + c(1-\varepsilon)^2 n + \dots$$
$$\leq c \cdot n \underbrace{\left(1 + (1-\varepsilon) + (1-\varepsilon)^2 + \dots\right)}_{\text{converging geometric series}}$$
$$\leq c \cdot n \, 1/\varepsilon$$

A randomly chosen pivot will be "in the middle" often enough to cause the problem size to decrease rapidly.

Suppose we want a pivot that has $\leq 3/4\, n$ elements smaller
and $\leq 3/4\, n$ elements larger



Probability that we choose a good pivot is 1/2

$\Rightarrow$ Expected number of choices before we pick a good pivot is at most 2

Remember: Let X = # fair coin tosses before head (random variable)

expected value of X $\rightarrow$ $E[X] = \sum_{x=1}^{\infty} x \Pr[X=x]$ (definition)

$$= \sum_{x=1}^{\infty} x \left(\tfrac{1}{2}\right)^x = \tfrac{1}{2} + \tfrac{2}{4} + \tfrac{3}{8} + \tfrac{4}{16} + \dots$$

$$2E[X] = 1 + 2/2 + 3/4 + 4/8 + \ldots$$

$$2E[X] - E[X] = 1 + 1/2 + 1/4 + 1/8 + \ldots$$

$$= 2$$

$$\Rightarrow E[X] = 2$$

Notice that if we fail to choose a good pivot, the probability of choosing a good pivot in the next step increases.

Let X_i = time spent by RSelect when A is of size between $n(3/4)^i$ and $n(3/4)^{i+1}$

$Y = X_0 + X_1 + X_2 + \ldots$ is the running time of RSelect.

$E[X_i] \leq 2\,cn\,(3/4)^i$ — cn: time spent partitioning; $n(3/4)^i$: |A| is at most this

$$\Rightarrow E[Y] = E[X_0] + E[X_1] + E[X_2] + \ldots$$

$$\leq \sum_{i=0}^{\infty} 2cn(3/4)^i$$

$$= 8cn$$

linearity of expectation
$E[A+B] = E[A] + E[B]$

+ foolproof, expected running time = $O(n)$

– randomized ... sometimes RSelect might take a long time.

Solution 4 — Deterministic Linear time k-Select

Idea Deterministically pick a good pivot p.

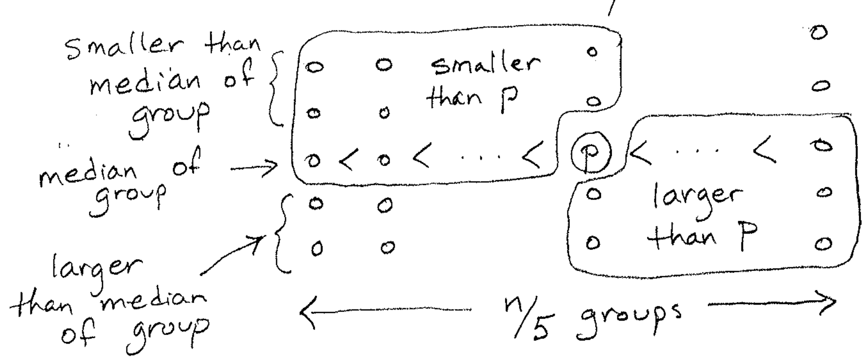
BSelect (A, k) — by Blum, Floyd, Pratt, Rivest, Tarjan 1972

```
BSelect(A, k)
    If |A| = 1 return A[1]
    p = GoodPivot(A)
    S = {A[i] | A[i] < p}
    L = {A[i] | A[i] > p}
    If |S| ≥ k return BSelect(S, k)
    else if |S| = k-1 return p
    else return BSelect(L, k - |S| - 1)
```

Good Pivot (A)

1. Divide A into $n/5$ groups of 5 elements each.
2. Find median of each group of 5.
3. Use BSelect to find the median, p, of the $n/5$ medians from step 2
4. return p.

At least how many elements of A are $< p$?



Arrange elements of A into this picture. Each element is a o.

Note: the algorithm doesn't do this. We do it to see how many elements are $< p$.

$\#$elts. smaller than $p \geq 3(n/10) - 1 \Rightarrow |S| \geq \frac{3n}{10} - 1 \Rightarrow |L| \leq 7n/10$

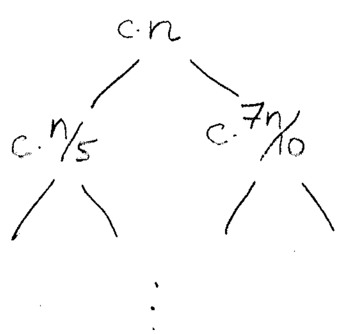
$\#$elts. larger than $p \geq 3(n/10) - 1 \Rightarrow |L| \geq \frac{3n}{10} - 1 \Rightarrow |S| \leq \frac{7n}{10}$

Running time of BSelect: $T(n)$

- pick good pivot (medians of $n/5$ groups of size 5 $= O(n)$
  + BSelect median of $n/5$ medians $= T(n/5)$
- partition $= O(n)$
- recurse $= O(7n/10)$

$$\Rightarrow \boxed{T(n) \leq c \cdot n + T(n/5) + T(7n/10)}$$

Recursion Tree for $T(n) = c \cdot n + T(n/5) + T(7n/10)$



$c \cdot n$

$c \cdot \frac{9}{10} n$

$c \cdot \left(\frac{9}{10}\right)^2 n$

$$\text{Total} \le c \cdot n \sum_{i=0}^{\log_{10/7} n} \left(\frac{9}{10}\right)^i \le c \cdot n \sum_{i=0}^{\infty} \left(\frac{9}{10}\right)^i = 10cn$$