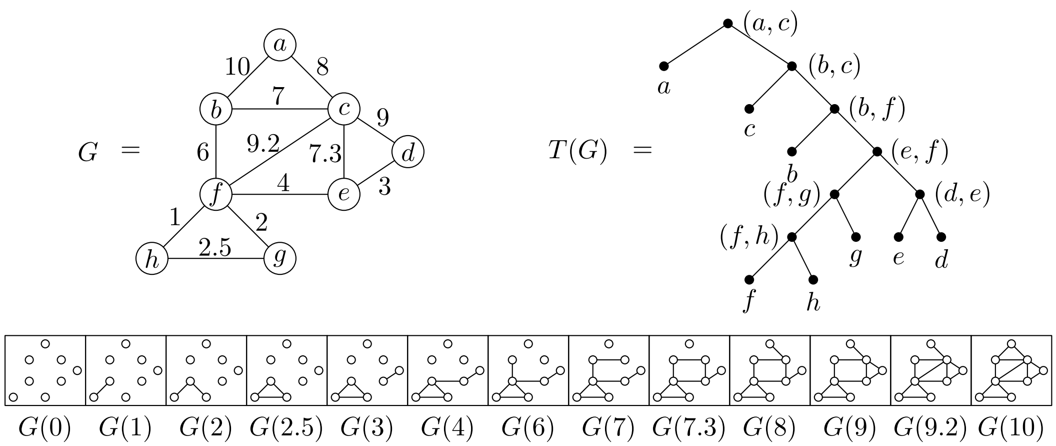
CPSC 320 Problem Set 5 23 October 2009
W. Evans Due: 30 October 2009 at the start of class.

1. Prove that any connected graph with n vertices and $n-1$ edges has no cycles.

2. Prove that any two minimum spanning trees of a graph G have exactly the same number of edges of any given weight. Use the edge swap idea from class.

   Prove that if all the edge weights of a graph are distinct then its minimum spanning tree is unique.

3. Let $G=(V,E)$ be a connected graph with edge weight $w(e)\in\mathbb{R}^+$ associated with each edge $e\in E$. Let $G(t)=(V,E(t))$ where $E(t)=\{e : w(e)\le t\}$. In other words, $G(t)$ is the subgraph of G that contains only those edges with weight at most t. Initially, when $t=0$, $G(0)$ contains no edges and has n connected components (the n vertices of G). As t increases from 0, the graph $G(t)$ aquires more and more edges and the number of connected components in $G(t)$ decreases until $G(t)=G$ and it contains just one connected component, which happens when $t=\max_{e\in E} w(e)$. We can picture this process as a binary, rooted tree $T(G)$ with n leaves. Each vertex of $T(G)$ represents a connected component of $G(t)$ for some value of t. The n leaves represent the n connected components of $G(0)$. If, as t increases, two connected components of $G(t)$ become connected (by including an edge e with weight $w(e)=t$), then the two vertices of $T(G)$ representing those two connected components are the children of the internal vertex in $T(G)$ representing the union of those components. (We label this internal vertex with the edge e.) For example:

   The tree $T(G)$ is called the *decomposition tree* of G.

   (a) Prove that the decomposition tree of G is the same as the decomposition tree of MST(G), where MST(G) is the minimum spanning tree of G.

   (b) Describe how to construct $T(G)$ from a given edge-weighted graph G in $O(m \lg n)$ time where $n=|V|$ and $m=|E|$.

   Hint: What additional information would be useful to store in the disjoint sets structure?