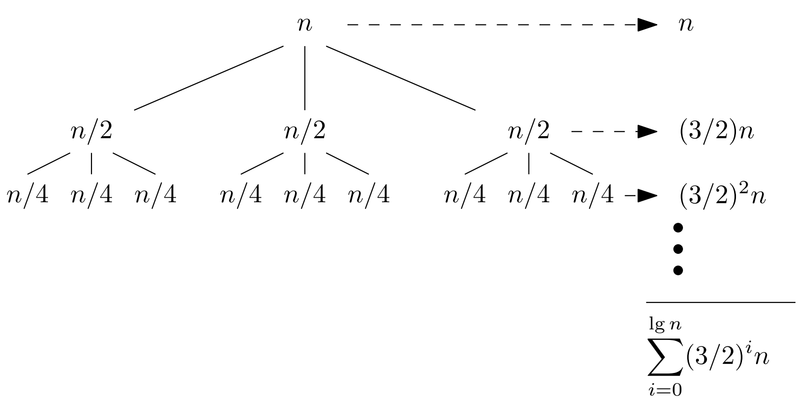
CPSC 320 Problem Set 2 solutions 5 October 2009

1. We know from class that when n is a power of 2, $T(n) = n(c \lg n + 1)$ and that $T(n)$ is an increasing function. Thus,

$$
\begin{aligned}
T(n) &\geq T(2^{\lfloor \lg n \rfloor}) && T(n) \text{ increasing} \\
&= 2^{\lfloor \lg n \rfloor}(c\lfloor \lg n \rfloor + 1) && n \text{ a power of 2} \\
&\geq 2^{\lg n - 1}(c(\lg n - 1) + 1) \\
&= (n/2)(c \lg n - c + 1) \\
&\geq (n/2)(c \lg n - c) \\
&= (n/4)c \lg n + (n/4)c \lg n - (n/2)c \\
&\geq (n/4)c \lg n && \text{for } n \geq 4 \\
&\geq dn \log n && \text{for } d < c/4
\end{aligned}
$$

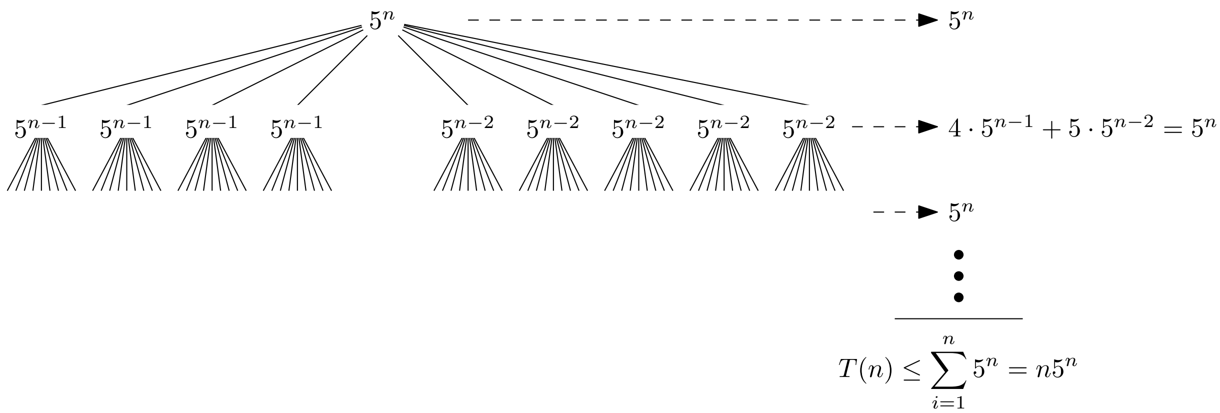
2.



The final summation is $\sum_{i=0}^{\lg n}(3/2)^i n = n((3/2)^{\lg n+1} - 1)/(3/2 - 1) = 3n^{\lg 3} - 2n$. We can avoid splitting our proof into cases if we prove the slightly weaker claim that $T(n) \leq 3n^{\lg 3} - 4n$. The inductive step is:

$$
\begin{aligned}
T(n) &= 3T(\lfloor n/2 \rfloor) + n \\
&\leq 3(3\lfloor n/2 \rfloor^{\lg 3} - 4\lfloor n/2 \rfloor) + n && \text{by inductive hypothesis} \\
&\leq 3(3(n/2)^{\lg 3} - 4(n-1)/2) + n && (n-1)/2 \leq \lfloor n/2 \rfloor \leq n/2 \\
&\leq 3n^{\lg 3} - 6(n-1) + n \\
&\leq 3n^{\lg 3} - 4n && \text{for } n \geq 6
\end{aligned}
$$

3.



Theorem. $T(n) \leq n5^n$ for all $n \geq 0$.

Proof. (by induction on n) Base case: For $n = 0$, $T(0) = 0 \leq 0{\cdot}5^0$. For $n = 1$, $T(1) = 1 \leq 1{\cdot}5^1$. Assume the claim is true for $T(n-1)$ and $T(n-2)$. We will show it is true for $T(n)$ where $n \geq 2$.

$$\begin{aligned}
T(n) &= 4T(n-1) + 5T(n-2) + 5^n \\
&\leq 4(n-1)5^{n-1} + 5(n-2)5^{n-2} + 5^n \qquad \text{by inductive hypothesis} \\
&= 4(n-1)5^{n-1} + (n-2)5^{n-1} + 5^n \\
&\leq 5(n-1)5^{n-1} + 5^n \\
&= (n-1)5^n + 5^n \\
&= n5^n.
\end{aligned}$$

□

4. $\lfloor n/2 \rfloor + 1$ of the integers from 0 to n have last bit equal to 0. The remaining $\lceil n/2 \rceil$ integers have last bit equal to 1. If one number is missing then we can determine, by examining the n last bits, what the last bit of the missing number is. We can then reduce the problem to looking for the missing number among the at most $n/2$ numbers with that last bit. (Continue the process by looking at the second to last bit, etc.)

   The running time is $T(n) \leq n + T(n/2)$ with $T(1) = 1$. Thus $T(n) = O(n)$.

5. The first recurrence, $T(n)$, has $a = 7$, $b = 2$, and $f(n) = n^2$. Since $n^2 \in O(n^{\log_2 7 - \epsilon})$ for $\epsilon > 0$ ($\log_2 7 = 2.8...$), the Master Theorem tells us that $T(n) \in \Theta(n^{\lg 7})$.

   The second recurrence, $T'(n)$, has $a = a$, $b = 4$, and $f(n) = n^2$. If $a = 49$ then $n^{\log_4 a} = n^{\log_2 7}$ and $T'(n) \in \Theta(n^{\lg 7})$. So choose $a = 48$.