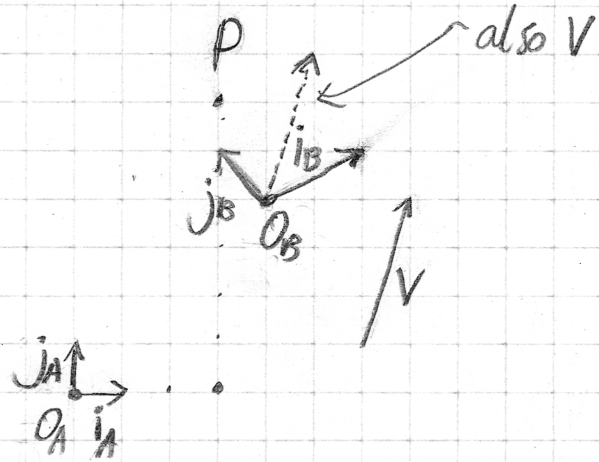
Example: Transformations as a change of basis



Express P and V with respect to F_A and F_B.

Develop the transformation matrix that takes points from F_A to F_B.

First: $P_A = O_A + 3\,i_A + 6\,j_A$ by inspection because i_A and j_A directly correspond to the grid cells.

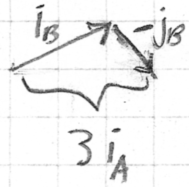
$V_A = i_A + 3 j_A \Rightarrow$ Note that the origin, O_A, is not involved for vectors.

$\Rightarrow P_A(3,6),\ V_A\langle 1,3\rangle$

Second: F_B is more difficult to deal with.
It is simplest to first be able to develop expressions for i_A and j_A in terms of i_B and j_B

We can note that $i_B - j_B$ takes us 3 grid cells to the right.

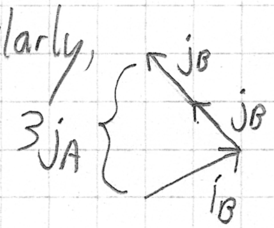
i.e.,



$3 i_A = i_B - j_B$

$\Rightarrow \boxed{i_A = \dfrac{i_B - j_B}{3}}$

Similarly,



$3 j_A = i_B + 2 j_B$

$\boxed{j_A = \dfrac{i_B + 2 j_B}{3}}$

We can now more easily express P_B and V_B:

$$P_B = O_B - i_A + 2 j_A \qquad \text{Takes advantage of grid.}$$

then substitute: $$= O_B - \frac{(i_B - j_B)}{3} + 2\,\frac{(i_B + 2j_B)}{3}$$

$$P_B = O_B + \frac{1}{3} i_B + \frac{5}{3} j_B \qquad \boxed{P_B\left(\tfrac{1}{3}, \tfrac{5}{3}\right)}$$

Similarly, $$V = i_A + 3 j_B$$

$$= \frac{(i_B - j_B)}{3} + 3\,\frac{(i_B + 2j_B)}{3}$$

$$= \frac{4}{3} i_B + \frac{5}{3} j_B \qquad \boxed{V_B \left\langle \tfrac{4}{3}, \tfrac{5}{3} \right\rangle}$$

For the transformation matrix $M_{A \to B}$

we need to express O_A, i_A, and j_A in terms of F_B.

$$O_A = O_B - 4 i_A - 4 j_A$$

$$= O_B - 4\,\frac{(i_B - j_B)}{3} - 4\,\frac{(i_B + 2j_B)}{3}$$

$$= O_B - \frac{8}{3} i_B - \frac{4}{3} j_B$$

$$= \left(-\tfrac{8}{3}, -\tfrac{4}{3}\right)_B$$

$$\begin{array}{cc} & \begin{array}{cccc} i_A & j_A & & O_A \end{array} \\ M_{A \to B} = & \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & -\frac{8}{3} \\ -\frac{1}{3} & \frac{2}{3} & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Check for P

$$\underbrace{\begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \\ 0 \\ 1 \end{bmatrix}}_{P_B} = \begin{bmatrix} 1+2-\frac{8}{3} \\ -1+4-\frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & -\frac{8}{3} \\ -\frac{1}{3} & \frac{2}{3} & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 3 \\ 6 \\ 0 \\ 1 \end{bmatrix}_A}_{P_A}$$

Check for V

$$\underbrace{\begin{bmatrix} \frac{4}{3} \\ \frac{5}{3} \\ 0 \\ 0 \end{bmatrix}}_{V_B} = \begin{bmatrix} \frac{1}{3}+1 \\ -\frac{1}{3}+2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & -\frac{8}{3} \\ -\frac{1}{3} & \frac{2}{3} & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}_A}_{V_A}$$

Note that vectors are transformed using $h = 0$