Ref

CPSC 314
Assignment 1

due: Wednesday, October 1, 2014, 11:59pm
Worth 9% of your final grade.

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Name: Solutions

Student Number:

Question 1		/ 15
Question 2		/ 9
Question 3		/ 9
Question 4		/ 5
Question 5		/ 4
Question 6		/ 8
Question 7		/ 50
TOTAL		/ 100

$3 j_A = i_c - j_c$

$j_A = \frac{i_c - j_c}{3}$

$3 i_A = j_c + 2 i_c$

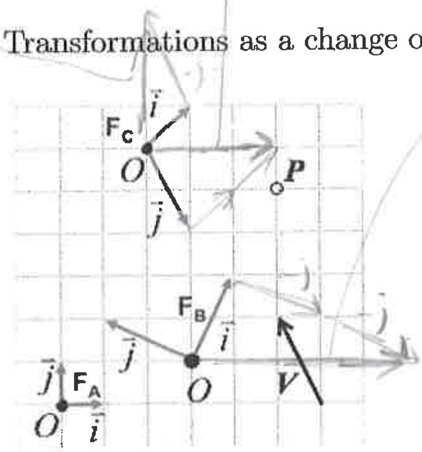
$\Rightarrow i_A = \frac{j_c + 2 i_c}{3}$

$5 i_A = i_B - 2 j_B$ and $5 j_A = j_B + 2 i_B$

$\Rightarrow i_A = \frac{i_B - 2 j_B}{5}$ $\quad j_A = \frac{j_B + 2 i_B}{5}$

CPSC 314 Assignment 1 September 2014

1. Transformations as a change of coordinate frame



$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_C$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_B = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & -1 \\ -\frac{2}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_A$$

$\boxed{P_A(5,5) \quad P_B(2,0) \quad P_C(\tfrac{5}{3}, \tfrac{4}{3})}$

$\boxed{V_A(-1,2) \quad V_B(\tfrac{3}{5}, \tfrac{4}{5}) \quad V_C(0,-1)}$

(a) (3 points) Express the coordinates of point P with respect to coordinate frames A, B, and C.

1 mark each

$P_A = O_A + 5\vec{i}_A + 5 j_A \Rightarrow P_A(5,5)$

$P_B = O_B + 2\vec{i}_B \Rightarrow P_B(2,0)$

$P_c(\tfrac{5}{3}, \tfrac{4}{3})$

$P_c = O_c + 3 i_A - j_A$
$= O_c + 3\left(\frac{j_c + 2i_c}{3}\right) - \left(\frac{-j_c + i_c}{3}\right)$
$= O_c + j_c + 2i_c + \frac{j_c}{3} - \frac{i_c}{3}$
$= O_c + \frac{4}{3} j_c + \frac{5}{3} i_c$

1 mark each

(b) (3 points) Express the coordinates of vector V with respect to coordinate frames A, B, and C.

$V_A = -i_A + 2 j_A \Rightarrow V_A(-1,2)$

$V_B = -i_A + 2 j_A$
$= -\left(\frac{i_B - 2 j_B}{5}\right) + 2\left(\frac{j_B + 2 i_B}{5}\right) \Rightarrow V_B(\tfrac{3}{5}, \tfrac{4}{5})$
$= \frac{3}{5} i_B + \frac{4}{5} j_B$

$V_C = -i_A + 2 j_A$
$= -\left(\frac{j_c + 2i_c}{3}\right) + 2\left(\frac{-j_c + i_c}{3}\right)$
$= -j_c$

$V_c(0,-1)$

(c) (3 points) Fill in the 2D transformation matrix that takes points from F_C to F_A, as given to the right of the above figure.

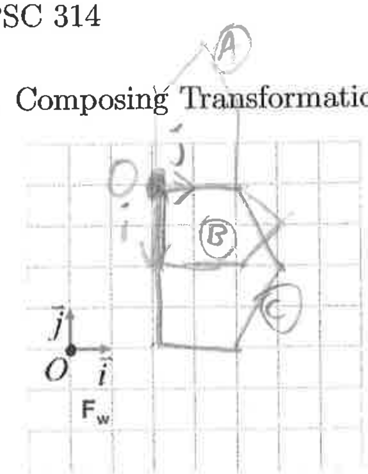
(d) (3 points) Fill in the 2D transformation matrix that takes points from F_A to F_B, as given to the right of the above figure.

(e) (3 points) Using the above two matrices, develop a 2D transformation matrix that takes points from F_C to F_B. Test your solution using point P.

$$= \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & -1 \\ -\frac{2}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix}_C$$

$$= \begin{bmatrix} \frac{3}{5} & -\frac{3}{5} & \frac{9}{5} \\ -\frac{1}{5} & -\frac{4}{5} & \frac{7}{5} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{4}{5} + \frac{9}{5} \\ -\frac{1}{3} - \frac{16}{15} + \frac{7}{5} \\ 1 \end{bmatrix}_B = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}_B$$

2. Composing Transformations



$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_w = \begin{array}{cccc} i & j & k & \text{origin} \\ \end{array} \begin{bmatrix} 0 & 1 & 0 & 2 \\ -2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{obj}$$

(a) (3 points) Consider a house in the xy-plane, defined by the coordinates (0,0), (2,0), (2,2), (1,3), 0,2).
Sketch the house after applying the following transformations. Assume that the matrix `m` is initialised to the identity matrix.

A `m.translate(2,4,0);`
B `m.rotate(-90, 0,0,1);`
C `m.scale(2,1,1);`

(b) (3 points) Give the resulting 4×4 transformation matrix. Assume that the transformation leaves z to be unaltered.

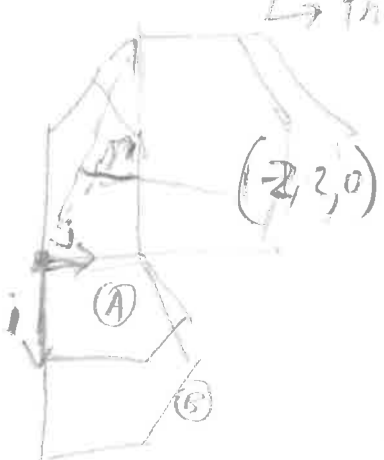
origin

$$\begin{bmatrix} i & j & k & o \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 2 \\ -2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) (3 points) What values would need to be assigned to `theta, sf, x, y, z` in order for the following transformations to yield an identical final transformation?

A `m.rotate(theta, 0,0,1);`
B `m.scale(a,b,c);`
C `m.translate(x,y,z);`

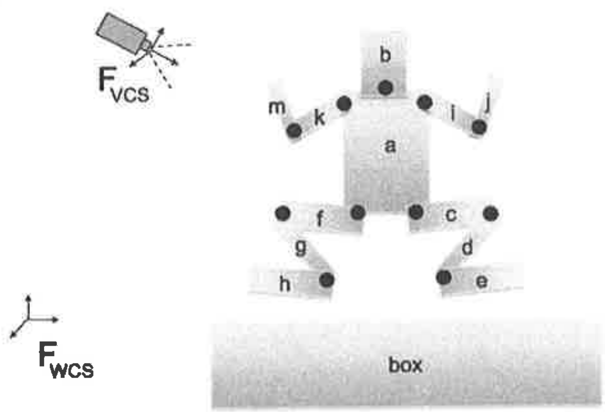
↳ this now translates in the rotated and scaled frame,

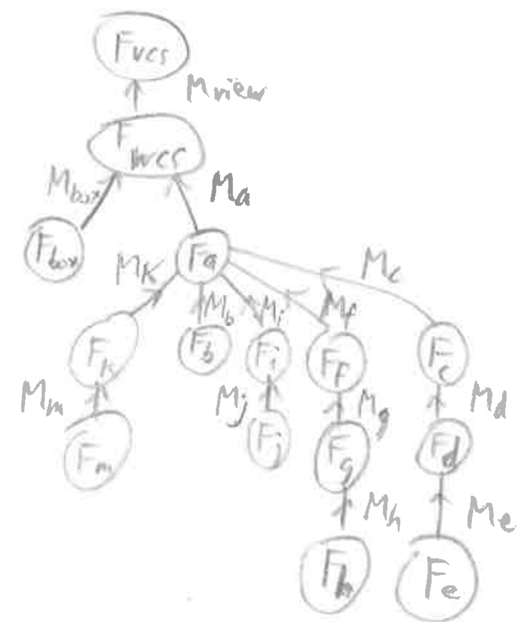


theta = $-90°$
a,b,c = (2,1,1)
x,y,z = (-2,2,0)

3. Scene Graphs

(a) (3 points) Draw a scene graph for the scene with the frog, shown below. Label each of the edges in the scene graph with a unique name, M_{name}, to represent the transformation matrix that takes points from its given frame to its parent frame. Assume that the body of the frog and the box are positioned relative to the world frame, and that the frames of all the other shown parts are defined relative to their proximal parent links, i.e., the links closer to the body.





(b) (2 points) Give an expression for the composite transformation that would be used when drawing link e i.e., to transform points in F_e to F_{VCS}. Your answer should be expressed as a product of the matrices used to label your scene graph.

$$P_{VCS} = M_{view} M_a M_c M_d M_e P_e$$

(c) (2 points) Give an expression for the composite transformation that transforms points in F_e to F_m. Your answer should be expressed as a product of the matrices used to label your scene graph.

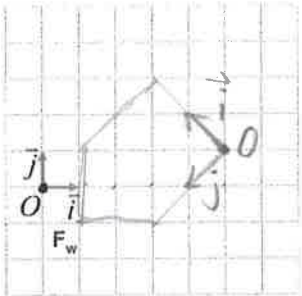
$$P_m' = M_m^{-1} M_k^{-1} M_c M_d M_e P_e$$

(d) (2 points) Describe how you could use the previous transformation to compute the distance, d, between the tip of the right foot and the tip of the left arm. Assume that the coordinate frames for each link are centred at the joints that connect them with their parent link, and assume that P_m and P_e define the tips of links m and e, respectively, in their local coordinate frame.

① Transform P_e to obtain P_m', as described above.

② $d = \| P_m - P_m' \|$

4. Transformations as a change of basis



$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_w = \begin{bmatrix} -1 & -1 & 0 & 5 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{obj}$$

(a) (2 points) Given the above transformation matrix, sketch the origin and basis vectors of the resulting object coordinate frame.

(b) (1 point) Sketch a house that is defined by the following points in the object coordinate frame: $P_1(0,0), P_2(2,0), P_3(2,2), P_4(1,3), P_5(0,2)$.

(c) (2 points) Give a sequence of translate(x,y,z), rotate(z,theta), and scale(a,b,c) transformations that would result in the transformation matrix given above.

translate$(5, 1, 0)$
rotate$(z, 135°)$
scale$(\sqrt{2}, \sqrt{2}, \sqrt{2})$

Note: this order can be swapped in the case of a uniform scale, ~~take~~ such as this case. However, a rotate after a non-uniform scale will not result in the desired effect.

5. (4 points) Viewing Transformations
Determine the viewing transformation, M_{view}, that takes points from WCS (world coordinates) to VCS (viewing or camera coordinates) for the following camera parameters: $P_{eye} = (-20, 30, -10), P_{ref} = (-20, 0, -10), V_{up} = (1, 0, 0)$. Do not bother with numerically inverting any matrices.

Build M_{cam} first:

$$M_{cam} = \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 & -20 \\ 0 & 0 & 1 & 30 \\ 1 & 0 & 0 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{cccc} i & j & k & \text{origin} \end{array} \end{array}$$

$M_{view} = M_{cam}^{-1}$

origin $= P_{eye} = (-20, 30, -10)$

$$\vec{k} = \frac{P_{eye} - P_{ref}}{\|P_{eye} - P_{ref}\|} = \frac{(0, 30, 0)}{\|(0, 30, 0)\|} = 0, 1, 0$$

$$\vec{i} = \frac{\vec{V_{up}} \times \vec{k}}{\|\vec{V_{up}} \times \vec{k}\|} = \langle 1, 0, 0 \rangle \times \langle 0, 1, 0 \rangle = \langle 0, 0, 1 \rangle$$

$$\vec{j} = \vec{k} \times \vec{i} = (0, 1, 0) \times (0, 0, 1) = (1, 0, 0)$$

6. Rotation Matrices

(a) (2 points) The columns of a rotation matrix have unit magnitude and they should all be orthogonal to each other, i.e., have a zero dot product. Show that the inverse of a rotation matrix is given by its transpose.

Define $R = \begin{bmatrix} a & b & c \end{bmatrix}$

$R^T R = I = R^{-1} R$

Then $R^T R = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} a \cdot a & b \cdot a & c \cdot a \\ a \cdot b & b \cdot b & c \cdot b \\ a \cdot c & b \cdot c & c \cdot c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) (6 points) In order for a rotation matrix to represent a rigid body rotation, there is one more constraint that it should satisfy, in addition to those listed above. Which of the following 4×4 matrices represent a valide rotation matrix? Why or why not? What is the extra constraint that rotation matrices should satisfy?

$$A = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} & 1 & & \\ -1 & & & \\ & 1 & & \\ & & & 1 \end{bmatrix} \quad D = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

No A: This is a reflection of z, which is not a rotation.

Yes B: This is a reflection of x and y, which is the same as Rot(z, 180°)

$$\begin{bmatrix} \cos(180°) & -\sin(180°) & & \\ \sin(180°) & \cos(180°) & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

No C: The second column is not unit size, i.e., $\|\langle 1,0,1 \rangle\| = \sqrt{2}$

No D: This swaps x and y values, so it might seem like Rot(180°, <1,1,0>). But that would also result in $z' \leftarrow -z$. So this transformation mirrors about the plane $x = y$.

Rotation matrices should also have a determinant of +1. Or, equivalently, $\vec{i} \times \vec{j} = \vec{k}$