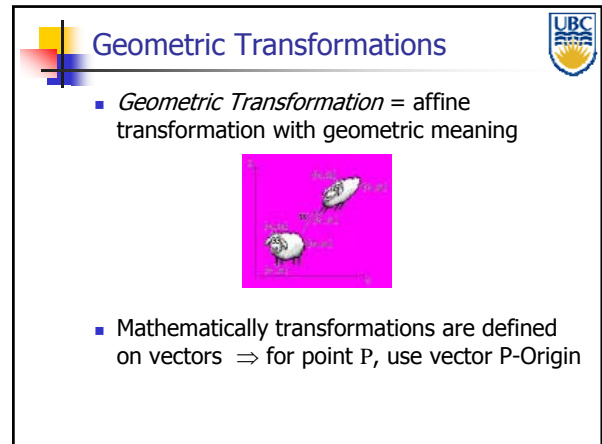




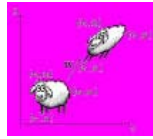
Chapter 3

Transformations

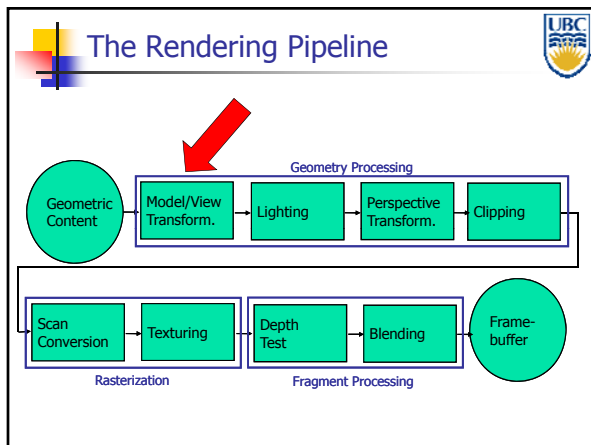
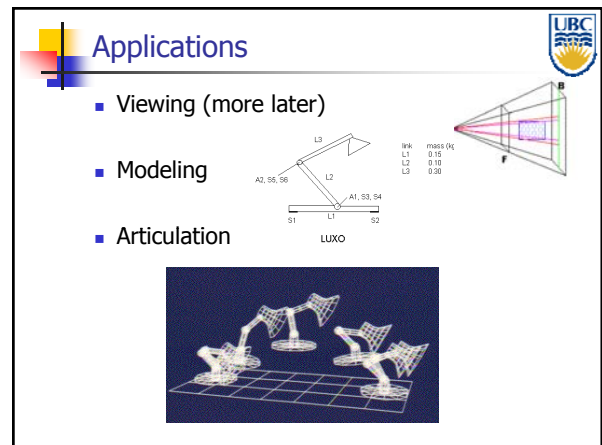



Geometric Transformations

- Geometric Transformation = affine transformation with geometric meaning

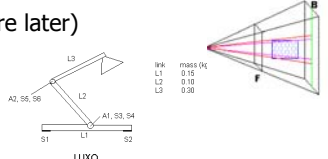
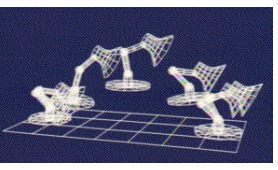
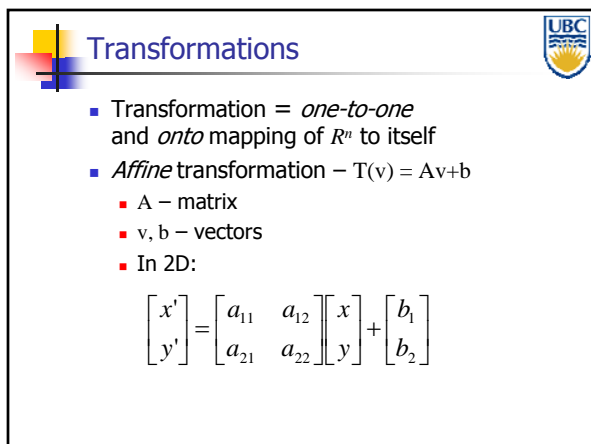


- Mathematically transformations are defined on vectors \Rightarrow for point P, use vector P-Origin

Applications

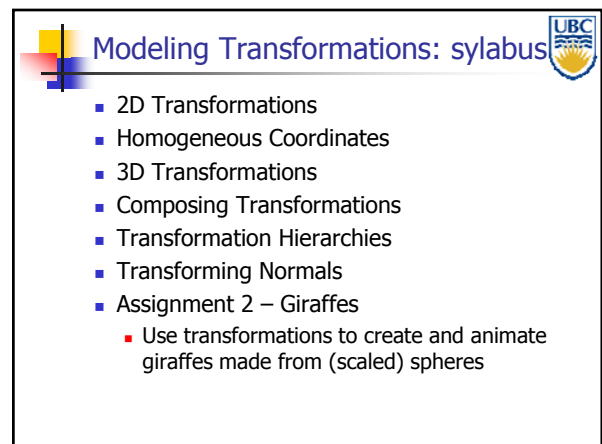
- Viewing (more later)
- Modeling
- Articulation

Transformations

- Transformation = *one-to-one* and *onto* mapping of R^n to itself
- Affine transformation – $T(v) = Av+b$
 - A – matrix
 - v, b – vectors
 - In 2D:


$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$



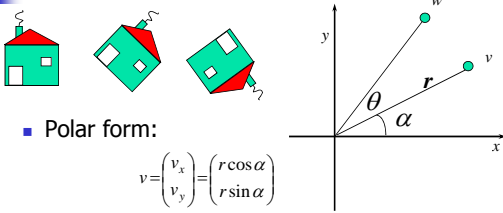
Modeling Transformations: syllabus

- 2D Transformations
- Homogeneous Coordinates
- 3D Transformations
- Composing Transformations
- Transformation Hierarchies
- Transforming Normals
- Assignment 2 – Giraffes
 - Use transformations to create and animate giraffes made from (scaled) spheres

Transformations

- Transforming an object = transforming all its points
- Transforming a polygon = transforming its vertices
- Why?
 

Rotation (using high school trigo...)




- Polar form:

$$v = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix}$$
- Rotating v counterclockwise by θ to w :

$$w = \begin{pmatrix} w_x \\ w_y \end{pmatrix} = \begin{pmatrix} r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \end{pmatrix} = \begin{pmatrix} r \cos \alpha \cos \theta - r \sin \alpha \sin \theta \\ r \cos \alpha \sin \theta + r \sin \alpha \cos \theta \end{pmatrix}$$

Scaling

- $v = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ – vector in XY plane
- Scaling operator S with parameters (s_x, s_y) :

$$S^{(s_x, s_y)}(V) = \begin{pmatrix} s_x v_x \\ s_y v_y \end{pmatrix}$$


Rotation

- Matrix form:

$$w = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} r \cos \alpha \\ r \sin \alpha \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} v$$
- Rotation operator R (at the origin) with parameter θ :

$$R^\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Scaling

- Matrix form:

$$S^{(s_x, s_y)}(V) = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} s_x v_x \\ s_y v_y \end{pmatrix}$$
- Independent in x and y

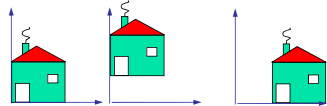
Rotation Properties

- R^0 is orthonormal

$$(R^\theta)^{-1} = (R^\theta)^T$$
- $R^{-\theta}$ - rotation by $-\theta$ is


$$R^{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = (R^\theta)^{-1}$$

Translation



- Translation operator T with parameters (t_x, t_y) :

$$T^{(t_x, t_y)}(v) = \begin{pmatrix} v_x + t_x \\ v_y + t_y \end{pmatrix}$$
- How can we write this in matrix form?



Translation

- Using homogeneous coordinates, translation operator may be expressed as:

$$T^{(t_x, t_y)}(v^h) = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix} = \begin{pmatrix} v_x + t_x \\ v_y + t_y \\ 1 \end{pmatrix}$$

Translation: Homogeneous Coordinates

- To represent T in matrix form – introduce homogeneous coordinates:

$$v = \begin{pmatrix} v_x \\ v_y \end{pmatrix} \rightarrow v^h = \begin{pmatrix} v_x^h \\ v_y^h \\ v_w^h \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}$$

Homogeneous Coordinates

$$\text{Rotation} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Scale} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Other ideas for uniform scale?

Translation: Homogeneous Coordinates

- Conversion (projection) from homogeneous space to Euclidean:

$$v = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_x^h / v_w^h \\ v_y^h / v_w^h \end{pmatrix}$$
- Projections is not 1:1

$$\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0.5 \end{pmatrix} \text{ all project to } \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Affine Transformations

- Combining transformations

$$ST(v) = C(Av + b) + d = (CA)v + (Cb + d)$$
- Same format (multiply by matrix & add vector)

$$A' = CA \quad b' = Cb + d$$
- Same holds in homogeneous coordinates – becomes simply matrix multiplication...

Transformation Composition

- What operation rotates XY by θ around $P = \begin{pmatrix} p_x \\ p_y \end{pmatrix}$?
- Answer:
 - Translate P to origin
 - Rotate around origin by θ
 - Translate back

Compositing of Affine Transformations

- Two different interpretations of composite:
 - 1) read from inside-out as transformation of object
 - 1a) Translate object by $-t$
 - 1b) Rotate object by Φ
 - 1c) Translate object by t
 - 2) read from outside-in as transformation of the coordinate frame
 - 2c) Translate frame by t
 - 2b) Rotate frame by $-\Phi$ (i.e. rotate object by Φ)
 - 2a) Translate frame by $-t$

Transformation Composition

$$T^{(p_x, p_y)} R^\theta T^{(-p_x, -p_y)} (V)$$

$$= \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -p_x \\ 0 & 1 & -p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix}$$

Compositing of Affine Transformations


- Example scene:

Compositing of Affine Transformations

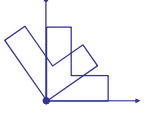
- In general:
 - Transform geometry into coordinate system where operation becomes simpler
 - Perform operation
 - Transform geometry back to original coordinate system


Compositing of Affine Transformations

- First Interpretation:
 - Step 1: translate object by $-t$ (move to origin)

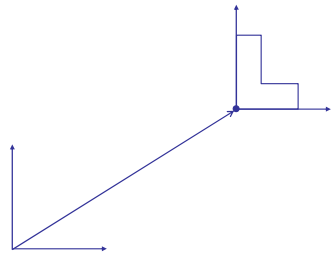
Compositing of Affine Transformations 


- First Interpretation:
 - Step 2: rotate object by Φ



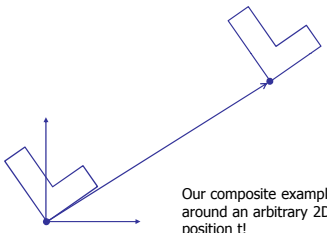
Compositing of Affine Transformations 

- Second interpretation:
 - Step 1: translate frame (move origin to object)




Compositing of Affine Transformations 

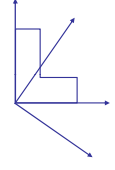
- First Interpretation:
 - Step 3: translate object by t (move back)




Our composite example is a rotation around an arbitrary 2D point with position t !

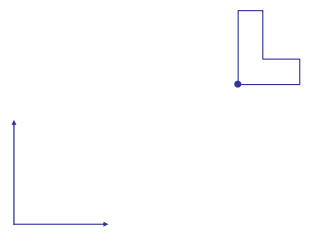
Compositing of Affine Transformations 


- Second interpretation:
 - Step 2: rotate frame by $-\Phi$ (i.e rotate obj. by Φ)



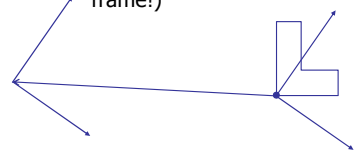
Compositing of Affine Transformations 

- Example scene, second interpretation:




Compositing of Affine Transformations 

- Second interpretation:
 - Step 3: translate frame back (vector $-t$ in new frame!)



Transformations Quiz




- What do these transformations do?


$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$


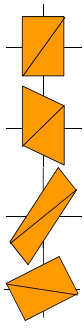
Shear & Mirroring/Reflection




- Shear (canonical)

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
- Mirroring/Reflection

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
- What is the relation between shears and rotations?




Transformations Quiz




- And these homogeneous ones?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$


$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$


Linear Transformations





- Combinations of
 - shear
 - scale
 - rotate
 - reflect
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{matrix} x' = ax + by \\ y' = cx + dy \end{matrix}$$
- Properties (why?)
 - satisfies $T(sx+ty) = s T(x) + t T(y)$
 - origin maps to origin
 - Straight lines map to straight lines
 - parallel lines remain parallel
 - closed under composition


Transformations Quiz



- How to mirror through arbitrary line in XY?
- What transformation achieves this?

Affine Transformations



- Combinations of
 - linear transformations
 - translations
$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$
- Properties (why?)
 - origin does not necessarily map to origin
 - lines map to lines
 - parallel lines remain parallel
 - ratios are preserved
 - closed under composition

3D Transformations

- All 2D transformations extend to 3D
- In homogeneous coordinates:


Scaling Translation Rotation around the z axis

$$S^{(s_x, s_y, s_z)} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{(t_x, t_y, t_z)} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z^\theta = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

glScalef(a,b,c); glTranslatef(a,b,c); glRotatef(angle,0,0,1);



Arbitrary Rotation

- Proof:

$$R(X) = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = U$$

- Similarly $R(Y) = V$ & $R(Z) = W$

3D Rotation in X, Y

around x axis: around y axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

glRotatef(angle,1,0,0); glRotatef(angle,0,1,0);

- general OpenGL command

glRotatef(angle,x,y,z);

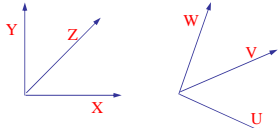
Arbitrary Rotation (cont.)

- Inverse (=transpose) transformation R^{-1} provides mapping from UVW to XYZ
- E.g.

$$R^{-1}(U) = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_x^2 + u_y^2 + u_z^2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = X$$

- Comment: Used for mapping between XY and arbitrary plane

Arbitrary Rotation



- Problem:
 - Given two orthonormal coordinate systems XYZ and UVW
 - Find transformation from one to the other
- Answer:
 - Transformation matrix R whose columns are U, V, W :

$$R = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix}$$

3D Shear

- shear in x

$$xshear(sy, sz) = \begin{bmatrix} 1 & sy & sz & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- shear in y

$$yshear(sx, sz) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ sx & 1 & sz & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- shear in z

$$zshear(sx, sy) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ sx & sy & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Undoing Transformations: Inverses

$\mathbf{T}(x, y, z)^{-1} = \mathbf{T}(-x, -y, -z)$
 $\mathbf{T}(x, y, z) \mathbf{T}(-x, -y, -z) = \mathbf{I}$

$\mathbf{R}(z, \theta)^{-1} = \mathbf{R}(z, -\theta) = \mathbf{R}^T(z, \theta)$ (**R is orthogonal**)
 $\mathbf{R}(z, \theta) \mathbf{R}(z, -\theta) = \mathbf{I}$

$\mathbf{S}(sx, sy, sz)^{-1} = \mathbf{S}\left(\frac{1}{sx}, \frac{1}{sy}, \frac{1}{sz}\right)$
 $\mathbf{S}(sx, sy, sz) \mathbf{S}\left(\frac{1}{sx}, \frac{1}{sy}, \frac{1}{sz}\right) = \mathbf{I}$

Composing Transformations

- scaling

$$S_2 \bullet S_1 = \begin{bmatrix} sx_1 \cdot dx_2 & & & \\ & sy_1 \cdot sy_2 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

scales multiply
- rotation

$$R_2 \bullet R_1 = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) & & \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

rotations add

3D Transformations - Composition

- Questions:
 - Is $S_1 S_2 = S_2 S_1$?
 - Is $T_1 T_2 = T_2 T_1$?
 - Is $R_1 R_2 = R_2 R_1$?
 - Is $S_1 R_2 = R_2 S_1$?
 -

Composing Transformations

ORDER MATTERS!

$T_a T_b = T_b T_a$, but $R_a R_b \neq R_b R_a$ and $T_a R_b \neq R_b T_a$

Composing Translations

$T_1 = T(dx_1, dy_1) = \begin{bmatrix} 1 & dx_1 & & \\ & 1 & dy_1 & \\ & & 1 & \\ & & & 1 \end{bmatrix}$
 $T_2 = T(dx_2, dy_2) = \begin{bmatrix} 1 & dx_2 & & \\ & 1 & dy_2 & \\ & & 1 & \\ & & & 1 \end{bmatrix}$

$P'' = T_2 \bullet P' = T_2 \bullet [T_1 \bullet P] = [T_2 \bullet T_1] \bullet P$, where

$T_2 \bullet T_1 = \begin{bmatrix} 1 & dx_1 + dx_2 & & \\ & 1 & dy_1 + dy_2 & \\ & & 1 & \\ & & & 1 \end{bmatrix}$

Translations add

Another Transformations Quiz

- What does each transformation preserve?

	lines	parallel lines	distance	angles	normals	convexity
scaling						
rotation						
translation						
shear						