CPSC 314
Theory Assignment 1

Due in class, Thursday, September 22, 2011

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Name: Provided Solution ______________________________

Student Number: ______________________________

Question 1	/ 0
Question 2	/ 1
Question 3	/ 1
Question 4	/ 1
Question 5	/ 1
Question 6	/ 1
TOTAL	/ 5

1. (0 points) Print the plagiarism policy form from the web (http://www.ugrad.cs.ubc.ca/~ cs314/Vsep2011/plag.html). Sign it and submit with your assignment. **Assignments without the form, will not be checked**.

2. (1 point) Vectors

$$a = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \qquad b = \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix}$$

   - compute $a \cdot b$,

$$a \cdot b = 1 \cdot (-2) + (-2) \cdot 5 + 4 \cdot 3 = 0$$

   - compute $a^T b$
     Following the rules of matrix multiplication,

$$a^T b = \begin{pmatrix} 1 & -2 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} = a \cdot b = 0$$

   - compute $a \times b$
     One of the ways of calculating cross products is via a matrix determinant:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ -2 & 5 & 3 \end{vmatrix} = \mathbf{i} \cdot (-2 \cdot 3 - 4 \cdot 5) - \mathbf{j}(1 \cdot 3 - 4 \cdot (-2)) + \mathbf{k}(1 \cdot 5 - (-2) \cdot (-2)) =$$

$$= -26\mathbf{i} - 11\mathbf{j} + \mathbf{k} = \begin{pmatrix} -26 \\ -11 \\ 1 \end{pmatrix},$$

     where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ *are unit vectors.*

   - compute $b \times a$

$$b \times a = -a \times b = \begin{pmatrix} 26 \\ 11 \\ -1 \end{pmatrix}$$

3. (1 point) Matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ -4 & 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 1 & 6 \\ 1 & 2 & 2 \\ 1 & -1 & 2 \end{pmatrix}$$

- Compute $C = AB$.

$$\begin{pmatrix} 8 & 2 & 16 \\ 7 & -1 & 14 \\ -8 & -5 & -16 \end{pmatrix}$$

- Does $AB = BA$?
Though in some specific cases matrix multiplication is commutative, generally it is not. Here, if you calculate BA you will see that it is different from AB:

$$BA = \begin{pmatrix} -21 & 14 & 32 \\ -7 & 8 & 19 \\ -7 & 2 & 4 \end{pmatrix}$$

- Given the vector a from the previous question, compute $c = Aa$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ -4 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot (-2) + 3 \cdot 4 \\ 0 \cdot 1 + 2 \cdot (-2) + 5 \cdot 4 \\ (-4) \cdot 1 + 1 \cdot (-2) + 3 \cdot 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 16 \\ 6 \end{pmatrix}$$

- Given the vector a from the previous question, compute $d = a^T A$.

$$\begin{pmatrix} 1 & -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ -4 & 1 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \cdot 1 - 2 \cdot 0 + 4 \cdot (-4) & 1 \cdot 2 - 2 \cdot 2 + 4 \cdot 1 & 1 \cdot 3 - 2 \cdot 5 + 4 \cdot 3 \end{pmatrix} =$$

$$= \begin{pmatrix} -15 & 2 & 5 \end{pmatrix}$$

4. (1 point) Normals and Planes

   T is a triangle in 3D with vertices $P_1 = (1,1,0)$, $P_2 = (1,0,2)$ and $P_3 = (3,2,0)$ (counterclockwise around the normal).

   - Compute the normal to T.
     We know that cross product of two vectors in a plane is orthogonal to that plane. Therefore, we can use the normalized cross product as the unit normal to the plane:

     $$\overrightarrow{P_1P_2} = (0,-1,2); \overrightarrow{P_1P_3} = (2,1,0)$$

     $$\vec{n} = \frac{\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}}{||\overrightarrow{P_1P_2} \times \ \overrightarrow{P_1P_3}||} = (\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})^T$$

   - Compute the area of T.
     One of the formulas to calculate the area of the triangle is that it is equal to half of the cross product of the triangle's sides, which we have already computed for the previous answer:

     $$S_T = \frac{1}{2}||\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}|| = \sqrt{6}$$

   - Compute the implicit plane equation $Ax + By + Cz + D = 0$ for the plane that T lies in.
     The plane equation is not unique: we choose all the coefficients A, B, C, D up to a constant factor. We know that (A, B, C) is collinear to $\vec{n}$. For convenience, we multiply the normal by $\sqrt{6}$ to get rid of the ugly denominator, we can take

     $$(A, B, C) = (-1, 2, 1)$$

     To calculate D, we use the fact that the plane should go through each of P_1, P_2, P_3. Let's use, for example, P_1:

     $$-1 \cdot 1 + 2 \cdot 1 + 1 \cdot 0 + D = 0$$

     $$D = -1$$

5. (1 point) Segments and Lines

   Given two segments in 2D: S_1 from $(0,1)$ to $(-1,2)$ and S_2 from $(0,0)$ to $(2,2)$,

   - Does the point $P = (0,1)$ lie on S_1?
     Let's presume that $P = (P_x, P_y)$ lies on S_1 and denote endpoints of S_1 by A and B. Then $\overrightarrow{AP}$ is collinear to $\overrightarrow{S_1}$, i.e. $\overrightarrow{AP} = \alpha\overrightarrow{AB}$ for some $0 \le \alpha \le 1$. So, we would write system of equations like that:

     $$\{ P_x - 0 = \alpha \cdot (-1 - 0)$$

     $$\{ P_y - 1 = \alpha \cdot (2 - 1)$$

If P lies on the line connecting A and B, those two equations are linearly dependent (i.e. the same up to a constant factor), then there exists one α satisfying both of them. We calculate this alpha by solving either of the equations and check if it lies within the $[0,1]$ range - that means that the point is inside the segment S_1.

In this particular case, since $P = A$, so there's no need to solve a system to say yes, $P \in S_1$.

- Do S_1 and S_2 intersect? If yes, compute the intersection, if no, explain.
  Let's denote $S_1 = \overrightarrow{AB}$ and $S_2 = \overrightarrow{CD}$. As we know, any point on segment S_1 can be expressed as $A + \alpha \cdot \overrightarrow{AB}$ for some $\alpha \in [0,1]$. Similarly, any point on S_2 can be expressed as $C + \beta \cdot \overrightarrow{CD}$ for some $\beta \in [0,1]$. If those two segments intersect, there must exist one point $E = A + \alpha \cdot \overrightarrow{AB} = C + \beta \cdot \overrightarrow{CD}$. So, we can write down a system of equations:

$$\{\, A_x + \alpha \cdot (B_x - A_x) = C_x + \beta \cdot (D_x - C_x)$$

$$\{\, A_y + \alpha \cdot (B_y - A_y) = C_y + \beta \cdot (D_y - C_y),$$

  solve for α and β, and check if both of them are within $[0,1]$ range.

  In our case, the system looks like:

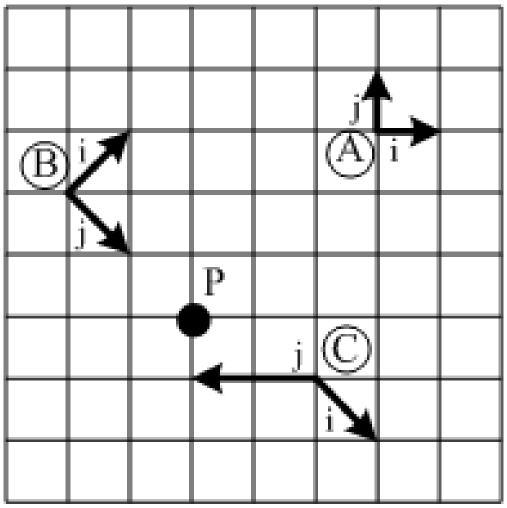
$$\{ -\alpha = 2\beta$$

$$\{ 1 + \alpha = 2\beta,$$

  So $\alpha = -0.5$ and $\beta = 0.25$. As $\alpha \notin [0,1]$, the final answer is no, those segment don't intersect.

6. (1 point) Frames

   Specify the coordinates of point P with respect to coordinate frames A, B and C.



We'll explain the method on the example B. Let's use coordinate system A as our reference system - we will express all the coordinates in A. Overall, we always can choose arbitrary convenient system of coordinates to do that. Then

$$\mathbf{i_B} = \mathbf{i_A} + \mathbf{j_A},$$

$$\mathbf{j_B} = \mathbf{i_A} - \mathbf{j_A},$$

$$O_B = O_A - 5\mathbf{i_A} - \mathbf{j_A},$$

where O_A and O_B are the origins of their respective coordinate frames. Also, $P = O_A - 3\mathbf{i_A} - 3\mathbf{j_A}$. Now, as we want to find out expression of P using unit vectors of B only, we should express $\mathbf{i_A}$ and $\mathbf{j_A}$ using $\mathbf{i_B}$,$\mathbf{j_B}$, thus finding coordinates of A's unit vector in B coordinate system. Using the previous equations,

$$\mathbf{i_A} = \frac{1}{2} \cdot (\mathbf{i_B} + \mathbf{j_B})$$

$$\mathbf{j_A} = \frac{1}{2} \cdot (\mathbf{i_B} - \mathbf{j_B}).$$

Also, $O_A = O_B + 5\mathbf{i_A} + \mathbf{j_A}$, so substituing it all to the expression for P, we get:

$$P = O_A - 3\mathbf{i_A} - 3\mathbf{j_A} = O_B + 5\mathbf{i_A} + \mathbf{j_A} - 3\mathbf{i_A} - 3\mathbf{j_A} =$$

$$= O_B + 2\mathbf{i_A} - 2\mathbf{j_A} = O_B + (\mathbf{i_B} + \mathbf{j_B}) - (\mathbf{i_B} - \mathbf{j_B}) = O_B + 2\mathbf{j_B}$$

So, $P_B = (0, 2)$.
Answers:
In coordinate frame A: $P_A = (-3, -3)$
In coordinate frame B: $P_B = (0, 2)$
In coordinate frame C: $P_C = (-1, 0.5)$