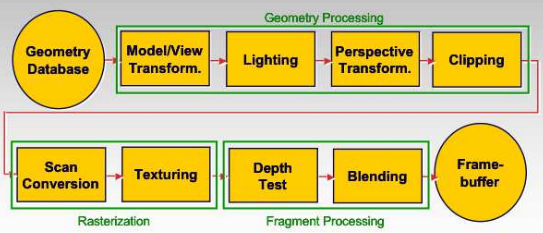
Affine Transformations

CPSC 314

The Rendering Pipeline



Modeling and Viewing Transformation

Affine transformations

- Linear transformations + translations
- Can be expressed as a *3x3* matrix + *3* vector

$$\mathbf{x}' = \mathbf{M} \cdot \mathbf{x} + \mathbf{t}$$

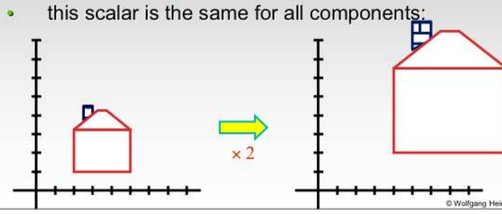
Scaling

Scaling

- a coordinate means multiplying each of its components by a scalar

Uniform scaling

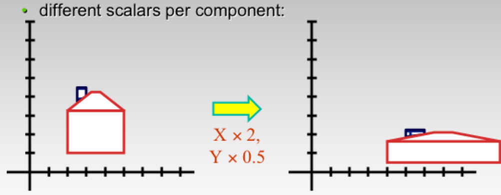
- this scalar is the same for all components:



Scaling

Non-uniform scaling:

- different scalars per component:



how can we represent this in matrix form?

Scaling (2D)

scaling operation: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$

or, in matrix form: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}}_{\textit{scaling matrix}} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

Scaling (3D)

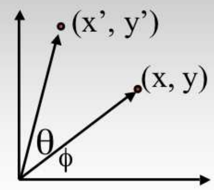
scaling operation:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} ax \\ by \\ cz \end{pmatrix}$$

or, in matrix form:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

2D Rotation From Trig Identities



$x = r\cos(\phi)$
$y = r\sin(\phi)$
$x' = r\cos(\phi + \theta)$
$y' = r\sin(\phi + \theta)$

Trig Identity…
$x' = r\cos(\phi)\cos(\theta) - r\sin(\phi)\sin(\theta)$
$y' = r\sin(\phi)\cos(\theta) + r\cos(\phi)\sin(\theta)$

Substitute…
$x' = x\,\mathbf{cos}(\theta) - y\,\mathbf{sin}(\theta)$
$y' = x\,\mathbf{sin}(\theta) + y\,\mathbf{cos}(\theta)$

2D Rotation Matrix

Easy to capture in matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Even though sin(θ) and cos(θ) are nonlinear functions of θ,

- x' is a linear combination of x and y
- y' is a linear combination of x and y

3D Rotation

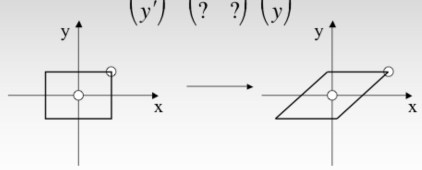
- About x axis: $$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
- About y axis: $$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
- About z axis: $$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Shear

Shear along x axis

- push points to right in proportion to height

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

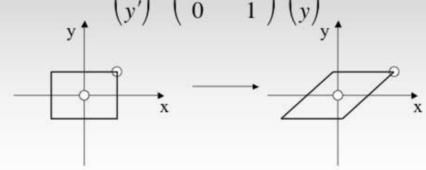


Shear

Shear along x axis

- push points to right in proportion to height

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & sh_x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

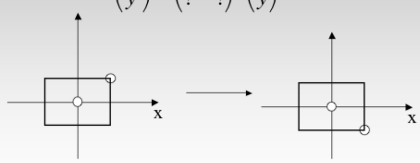


Reflection

Reflect across x axis

- Mirror

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

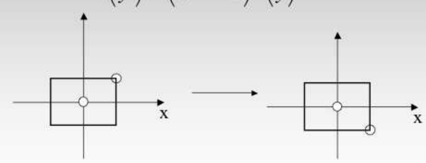


Reflection

Reflect across x axis

- Mirror

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

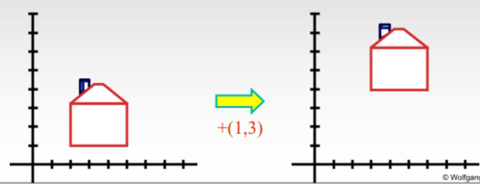


Affine Transformations

Translation:

- Add a constant (2D or 3D) vector:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$



Compositing of Linear and Affine Transformations

Example: 3D rotation around arbitrary axis

- Rotate axis to z-axis
- Rotate by ϕ around z-axis
- Rotate z-axis back to original axis
- Composite transformation:

$$\begin{aligned} R(v,\phi) &= R_z^{-1}(\alpha) \cdot R_y^{-1}(\beta) \cdot R_z(\phi) \cdot R_y(\beta) \cdot R_z(\alpha) \\ &= (R_y(\beta) \cdot R_z(\alpha))^{-1} \cdot R_z(\phi) \cdot (R_y(\beta) \cdot R_z(\alpha)) \end{aligned}$$

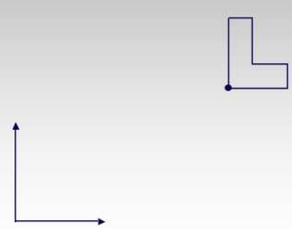
Compositing of Linear and Affine Transformations

In general:

- Transformation of geometry into coordinate system where operation becomes simpler
- Perform operation
- Transform geometry back to original coordinate system

Compositing of Affine Transformations

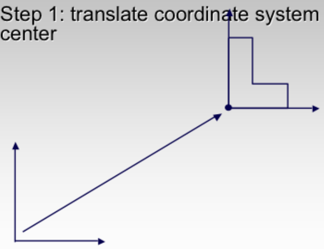
Example: Rotation around arbitrary center



Compositing of Affine Transformations

Example: Rotation around arbitrary center

- Step 1: translate coordinate system to rotation center



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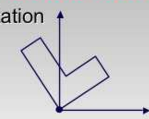
Compositing of Affine Transformations

Example: Rotation around arbitrary center

- Step 2: perform rotation



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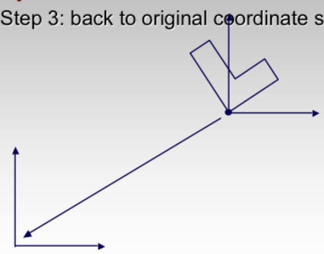
Compositing of Affine Transformations

Example: Rotation around arbitrary center

- Step 3: back to original coordinate system



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Compositing of Affine Transformations

Composite transformation:

- Is again an affine transformation

$$
\begin{aligned}
\mathbf{x}' &= \mathbf{Id}\cdot(R(\phi)\cdot(\mathbf{Id}\cdot\mathbf{x}+\mathbf{t}))-\mathbf{t}\\
&= \mathbf{Id}\cdot(R(\phi)\cdot\mathbf{x}+R(\phi)\cdot\mathbf{t})-\mathbf{t}\\
&= R(\phi)\cdot\mathbf{x}+(R(\phi)\cdot\mathbf{t}-\mathbf{t})\\
&= R(\phi)\cdot\mathbf{x}+\mathbf{t}'
\end{aligned}
$$

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Properties of Affine Transformations

Definition:

- A *linear combination* of points or vectors is given as

$$\mathbf{x}=\sum_{i=1}^{n} a_i\cdot\mathbf{x}_i,\ \text{for } a_i\in\Re$$

- An affine combination of points or vectors is given as

$$\mathbf{x}=\sum_{i=1}^{n} a_i\cdot\mathbf{x}_i,\ \text{with } \sum_{i=1}^{n} a_i=1$$

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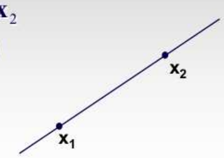
Properties of Affine Transformations

Example:

- Affine combination of 2 points

$$
\begin{aligned}
\mathbf{x} &= a_1\cdot\mathbf{x}_1+a_2\cdot\mathbf{x}_2,\ \text{with } a_1+a_2=1\\
&= (1-a_2)\cdot\mathbf{x}_1+a_2\cdot\mathbf{x}_2\\
&= \mathbf{x}_1+a_2\cdot(\mathbf{x}_2-\mathbf{x}_1)
\end{aligned}
$$



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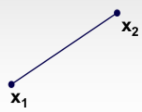
Properties of Affine Transformations

Definition:

- A convex combination is an affine combination where all the weights a_i are positive
- Note: this implies $0 \le a_i \le 1$, $i=1\ldots n$



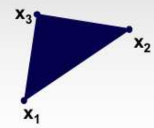
Properties of Affine Transformations

Example:

- Convex combination of 3 points

$$\mathbf{x} = \alpha \cdot \mathbf{x}_1 + \beta \cdot \mathbf{x}_2 + \gamma \cdot \mathbf{x}_3$$
$$\text{with } \alpha + \beta + \gamma = 1,\ 0 \le \alpha, \beta, \gamma \le 1$$

- α, β, and γ are called *barycentric coordinates*



Properties of Affine Transformations

Theorem:

- The following statements are synonymous
  - *A transformation T(x) is affine, i.e.:*

$$\mathbf{x}' = T(\mathbf{x}) := \mathbf{M} \cdot \mathbf{x} + \mathbf{t},$$
$$\text{for some matrix } \mathbf{M} \text{ and vector } \mathbf{t}$$

  - *T(x) preserves affine combinations, i.e.*

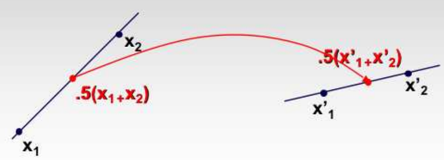
$$T(\sum_{i=1}^{n} a_i \cdot \mathbf{x}_i) = \sum_{i=1}^{n} a_i \cdot T(\mathbf{x}_i), \text{ for } \sum_{i=1}^{n} a_i = 1$$

  - *T(x) maps parallel lines to parallel lines*

Properties of Affine Transformations

Preservation of affine combinations:

- Can compute transformation of every point on line or triangle by simply transforming the *control points*



Homogeneous Coordinates

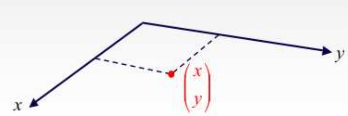
Homogeneous representation of points:

- Add an additional component $w=1$ to all *points*
- All multiples of this vector are considered to represent the same 3D point
- Use square brackets (rather than round ones) to denote homogeneous coordinates (different from text book!)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \equiv \begin{bmatrix} x \cdot w \\ y \cdot w \\ z \cdot w \\ w \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix}, \forall w \neq 0$$

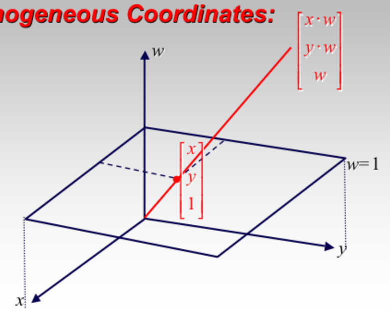
Geometrically In 2D

Cartesian Coordinates:



Geometrically In 2D

Homogeneous Coordinates:



Homogeneous Matrices

Affine Transformations

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & t_x \\ 0 & 0 & 0 & t_y \\ 0 & 0 & 0 & t_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Homogeneous Matrices

Combining the two matrices into one:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & t_x \\ 0 & 0 & 0 & t_y \\ 0 & 0 & 0 & t_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & t_x \\ m_{2,1} & m_{2,2} & m_{2,3} & t_y \\ m_{3,1} & m_{3,2} & m_{3,3} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Homogeneous Matrices

Note:

- Multiplication of the matrix with a constant does not change the transformation!

$$\tilde{T}\left(\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}\right) = \begin{bmatrix} m_{1,1}\cdot k & m_{1,2}\cdot k & m_{1,3}\cdot k & t_x\cdot k \\ m_{2,1}\cdot k & m_{2,2}\cdot k & m_{2,3}\cdot k & t_y\cdot k \\ m_{3,1}\cdot k & m_{3,2}\cdot k & m_{3,3}\cdot k & t_z\cdot k \\ 0 & 0 & 0 & k \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x'\cdot k \\ y'\cdot k \\ z'\cdot k \\ k \end{bmatrix}$$

$$\equiv \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}\right)$$

Homogeneous Vectors

Earlier discussion describes points only

- What about vectors (directions)?
- What is the affine transformation of a vector?
  - *Rotation*
  - *Scaling*
  - *Translation*

Vectors are invariant under translation!

Homogeneous Vectors

Representing vectors in homogeneous coordinates

- Need representation that is only affected by linear transformations, but not by translations
- This is achieved by setting $w=0$

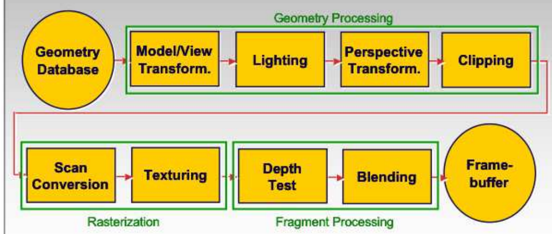
$$T\left(\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}\right) = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & t_x \\ m_{2,1} & m_{2,2} & m_{2,3} & t_y \\ m_{3,1} & m_{3,2} & m_{3,3} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 0 \end{bmatrix}$$

Homogeneous Coordinates

Properties

- Unified representation as 4-vector (in 3D) for
  - *Points*
  - *Vectors / directions*
- Affine transformations become 4x4 matrices
  - *Composing multiple affine transformations involves simply multiplying the matrices*
  - *3D affine transformations have 12 degrees of freedom*
    - Need mapping of 4 points to uniquely define transformation

The Rendering Pipeline



Modeling Transformation

Purpose:

- Map geometry from local *object coordinate system* into a global *world coordinate system*
- *Same as placing objects*

Transformations:

- Arbitrary affine transformations are possible
  - *Even more complex transformations may be desirable, but are not available in hardware*
    - Freeform deformations

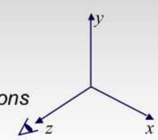
Viewing Transformation

Purpose:

- Map geometry from *world coordinate system* into *camera coordinate system*
- Camera coordinate system is *right-handed*, viewing direction is *negative* z-axis
- Same a placing camera

Transformations:

- Usually only *rigid body transformations*
  - *Rotations and translations*
- Objects have same size and shape in camera and world coordinates



Model/View Transformation

Combine modeling and viewing transform.

- Combine both into a single matrix
- Saves computation time if many points are to be transformed
- Possible because the viewing transformation directly follows the modeling transformation without intermediate operations

Homogeneous Planes And Normals

Planes in Cartesian Coordinates:

$$\{(x,y,z)^T \mid n_x x + n_y y + n_z z + d = 0\}$$

- n_x, n_y, n_z, and d are the parameters of the plane (normal and distance from origin)

Planes in Homogeneous Coordinates:

$$\{[x,y,z,w]^T \mid n_x x + n_y y + n_z z + dw = 0\}$$

Homogeneous Planes And Normals

Planes in homogeneous coordinates are represented as row vectors

- E=[n_x, n_y, n_z, d]
- Condition that a point $[x,y,z,w]^T$ is located in E

$$\begin{bmatrix}x\\y\\z\\w\end{bmatrix} \in E = [n_x, n_y, n_z, d] \Leftrightarrow [n_x, n_y, n_z, d] \cdot \begin{bmatrix}x\\y\\z\\w\end{bmatrix} = 0$$

Homogeneous Planes And Normals

Transformations of planes

$$[n_x, n_y, n_z, d] \cdot \begin{bmatrix}x\\y\\z\\w\end{bmatrix} = 0 \Leftrightarrow T([n_x, n_y, n_z, d]) \cdot (\mathbf{A} \cdot \begin{bmatrix}x\\y\\z\\w\end{bmatrix}) = 0$$

Homogeneous Planes And Normals

Transformations of planes

$$[n_x, n_y, n_z, d] \cdot \begin{bmatrix}x\\y\\z\\w\end{bmatrix} = 0 \Leftrightarrow ([n_x, n_y, n_z, d] \cdot \mathbf{A}^{-1}) \cdot (\mathbf{A} \cdot \begin{bmatrix}x\\y\\z\\w\end{bmatrix}) = 0$$

- Works for $T([n_x, n_y, n_z, d]) = [n_x, n_y, n_z, d]\mathbf{A}^{-1}$
- Thus: planes have to be transformed by the *inverse* of the affine transformation (multiplied from left as a row vector)!

Homogeneous Planes And Normals

Homogeneous Normals

- The plane definition also contains its normal
- Normal written as a vector $[n_x, n_y, n_z, 0]^T$

$$(\begin{bmatrix}n_x\\n_y\\n_z\\0\end{bmatrix} \cdot \begin{bmatrix}v_x\\v_y\\v_z\\0\end{bmatrix}) = 0 \Leftrightarrow ((\mathbf{A}^{-T} \cdot \begin{bmatrix}n_x\\n_y\\n_z\\0\end{bmatrix}) \cdot (\mathbf{A} \cdot \begin{bmatrix}v_x\\v_y\\v_z\\0\end{bmatrix})) = 0$$

- Thus: the normal to any surface has to be transformed by the inverse transpose of the affine transformation (multiplied from the right as a column vector)!

Transforming Homogeneous Normals

Inverse Transpose of

- Rotation by α
  - *Rotation by α*
- Scale by s
  - *Scale by 1/s*
- Translation by t
  - *Identity matrix!*
- Shear by a along x axis
  - *Shear by -a along y axis*

Coming Up…

Tuesday, Sep 18:

- OpenGL transformations, matrix stacks

Thursday, Sep 20:

- Cameras & perspective transformations