Curves and Surfaces

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- how to model curves?
- how to model surfaces ?

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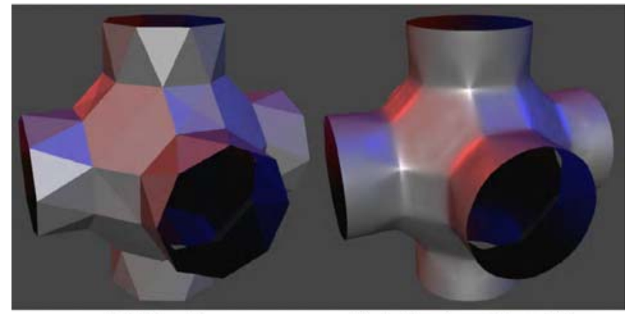
1. Subdivision curves and surfaces



Initial mesh   Butterfly scheme interpolation

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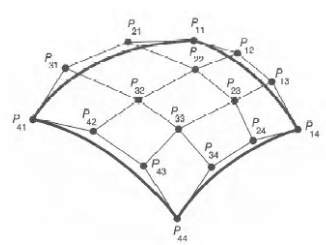
2. Parametric curves and surfaces



http://www.sjbaker.org/teapot/NewellTeaset.jpg

Parametric Curve examples

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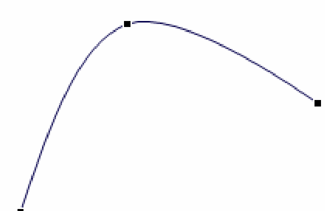
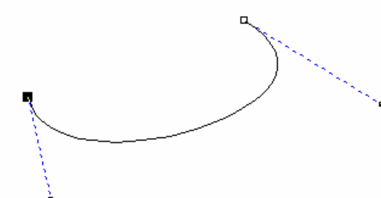


Powerpoint



CorelDraw

Test yourself...

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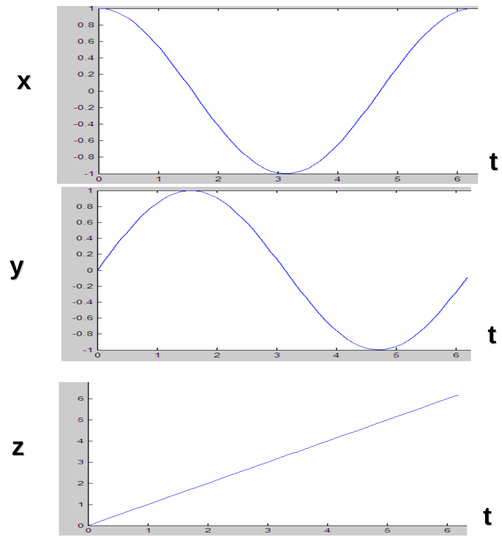

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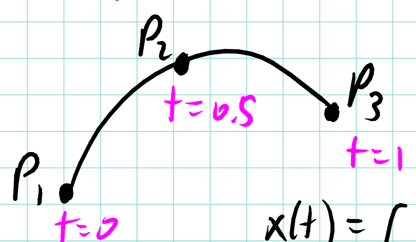
cubic parametric curves

$$x(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0$$
$$y(t) = b_3 t^3 + b_2 t^2 + b_1 t + b_0$$
$$z(t) = \ldots$$
$$t \in [0,1]$$

$$x(t) = \underbrace{\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}}_{A} = T \cdot A$$

Example with a Quadratic Parametric Curve



$$x(t) = [t^2 \; t \; 1] \cdot A$$

$$x_1 = [0 \quad 0 \quad 1] \cdot A$$
$$x_2 = [0.5^2 \;\; 0.5 \;\; 1] \cdot A$$
$$x_3 = [1 \quad 1 \quad 1] \cdot A$$

rewrite using a matrix

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0.5^2 & 0.5 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix}$$

$$G_x = C \cdot A$$

$$G = C \cdot A$$

coefficients of polynomial

constraint matrix

geometry vector, i.e., control points

We can now solve for A:

$$A = C^{-1} G_x$$

Now we can write out $x(t)$:

$$x(t) = T \cdot A = T C^{-1} G_x$$

$M \equiv C^{-1}$
$\equiv$ "basis matrix"

$$= T M G_x$$

$$x(t) = [t^2 \; t \; 1] \begin{bmatrix} 0 & 0 & 1 \\ 0.25 & 0.5 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

To render the curve:

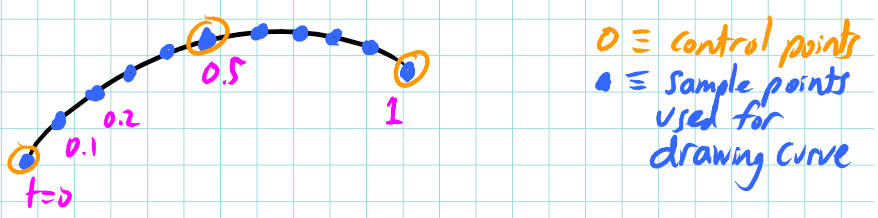
Compute a_2, a_1, a_0 : $A = M\,G_x$
compute b_2, b_1, b_0 : $B = M\,G_y$

for(t=0; t<=1; t+=Δt) {

$x = a_2 t^2 + a_1 t + a_0$
$y = b_2 t^2 + b_1 t + b_0$

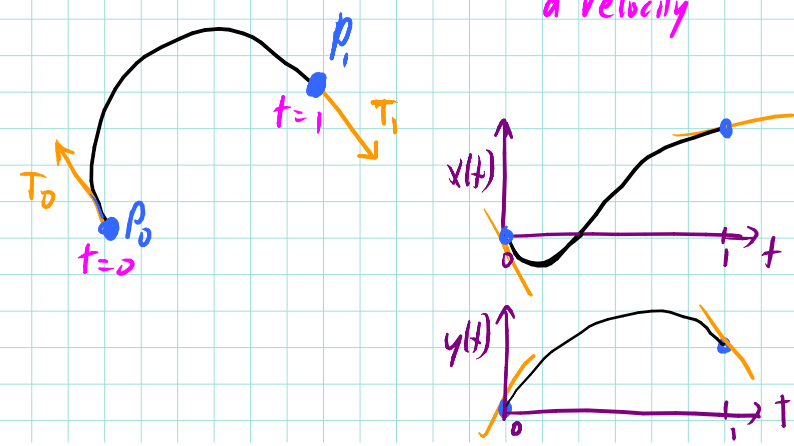
draw line from previous point to (x,y)

}



Hermite Curves

- parametric cubic curve
- defined by:
  - start, end points
  - start, end tangents → think of this as a velocity



$$x(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0$$
$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$$

$$\frac{dx}{dt} = x'(t) = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$$

$\left.\frac{dx}{dt}\right|_{t=0}$

$$x(0) = x_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} A$$
$$x(1) = x_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} A$$
$$x'(0) = dx/dt|_{t=0} = t_{0x} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} A$$
$$x'(1) = dx/dt|_{t=1} = t_{1x} = \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} A$$

$$\begin{bmatrix} x_0 \\ x_1 \\ t_{0x} \\ t_{1x} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$$

$$G_x = C A$$

$$\Rightarrow \quad A = C^{-1} G_x$$

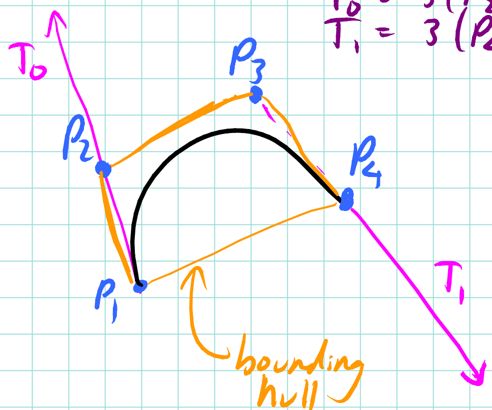
$$A = \underbrace{\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\text{Hermite basis matrix}} \begin{bmatrix} x_0 \\ x_1 \\ t_{x0} \\ t_{x1} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} & & \\ & M_H & \\ & & \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ t_{x0} \\ t_{x1} \end{bmatrix}$$

Bézier Curves

- variation on Hermite curves
- four points
- Hermite equivalent:

$$T_0 = 3(P_2 - P_1)$$
$$T_1 = 3(P_4 - P_3)$$



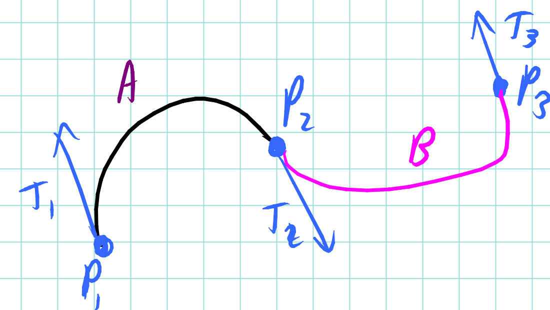
$$x(t) = [t^3 \; t^2 \; t \; 1] \begin{bmatrix} M_{bez} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$y(t) = T M_{bez} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

Note: can convert between curve types

$$x(t) = T M_H G_H = T M_{bez} G_{bez} \Rightarrow G_{bez} = M_{bez}^{-1} M_H G_H$$

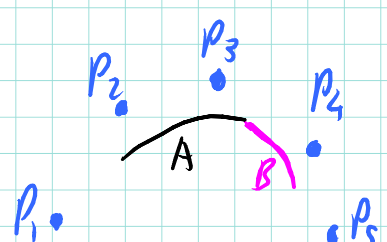
Piecewise Hermite and Bezier Curves



$$G_{A_x} = \begin{bmatrix} x_1 \\ x_2 \\ T_{1x} \\ T_{2x} \end{bmatrix} \qquad G_{B_x} = \begin{bmatrix} x_2 \\ x_3 \\ T_{2x} \\ T_{3x} \end{bmatrix}$$

Shared point and tangent vector

B-Spline Curves

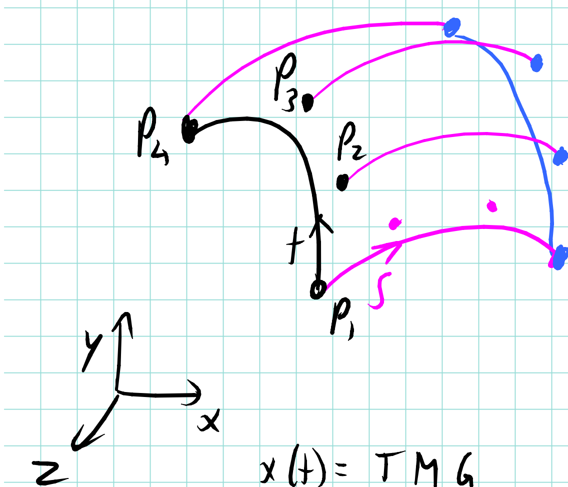


- Curve does not interpolate the control points
- but: it offers higher continuity between piecewise segments

  (2nd derivative "acceleration" is continuous)
- form the basis of "NURBS" Non-Uniform Rational B-Splines

$$G_{A_x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad G_{B_x} = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

From curves to surfaces



$$x(t) = T M G = T M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$[s^3 \; s^2 \; s \; 1]$

$$x_1(s) = S M G_{1x}$$
$$x_2(s) = S M G_{2x}$$
$$x_3(s) = S M G_{3x}$$
$$x_4(s) = S M G_{4x}$$

$\Downarrow$ equivalently

$$x_1(s) = G_{1x}^T M^T S^T = [\quad] \begin{bmatrix} M^T \end{bmatrix} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$

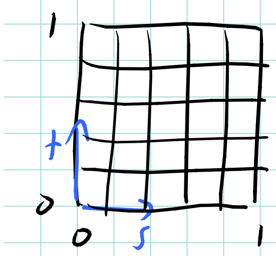
$$x(t) = T M \begin{bmatrix} G_{1x}^T M^T S^T \\ G_{2x}^T M^T S^T \\ G_{3x}^T M^T S^T \\ G_{4x}^T M^T S^T \end{bmatrix}$$

$$= T M \begin{bmatrix} \boxed{G_{1x}} \\ \boxed{} \\ \boxed{} \\ \boxed{G_{4x}} \end{bmatrix} M^T S^T$$

$$x(s,t) = T M G_x M^T S^T$$

geometry matrix
x coords of all 16 control points

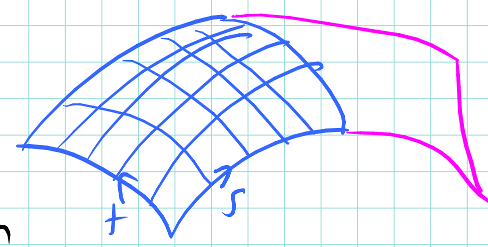
$$x(s,t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} G \end{bmatrix} \begin{bmatrix} M^T \end{bmatrix} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$

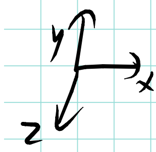


"parametric patch"

s, t

- user controls shape of patch by moving the control points
- difficult to compute analytic solution to ray-patch intersection





Rim:
{ 102, 103, 104, 105, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 }
Body:
{ 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27 }
{ 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40 }
Lid:
{ 96, 96, 96, 96, 97, 98, 99, 100, 101, 101, 101, 101, 0, 1, 2, 3 }
{ 0, 1, 2, 3, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117 }
Handle:
{ 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56 }
{ 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 28, 65, 66, 67 }
Spout:
{ 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83 }
{ 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95 }

Vertices:

{ 0.2000, 0.0000, 2.70000 }, { 0.2000, -0.1120, 2.70000 }, { 0.1120, -0.2000, 2.70000 },
{ 0.0000, -0.2000, 2.70000 }, { 1.3375, 0.0000, 2.53125 }, { 1.3375, -0.7490, 2.53125 },
{ 0.7490, -1.3375, 2.53125 }, { 0.0000, -1.3375, 2.53125 }, { 1.4375, 0.0000, 2.53125 },
{ 1.4375, -0.8050, 2.53125 }, { 0.8050, -1.4375, 2.53125 }, { 0.0000, -1.4375, 2.53125 },
{ 1.5000, 0.0000, 2.40000 }, { 1.5000, -0.8400, 2.40000 }, { 0.8400, -1.5000, 2.40000 },
etc.

[Google Utah teapot