CPSC 314
Quiz 1

2pm, Thursday October, 20 2005

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Name: ____________________

Student Number: ____________________

Question 1		/ 12
Question 2		/ 12
Question 3		/ 15
Question 4		/ 22
Question 5		/ 24
Question 6		/ 15
TOTAL		/ 100

1. Vectors

   (a) (6 points) Write an efficient test to check if two vectors v_1 and v_2 in 3D are *parallel* to one another.

   Check if $v_1 \times v_2 = 0$.

   (b) (6 points) Write an efficient test to check if two vectors v_1 and v_2 in 3D are *perpendicular* to one another.

   Check if $v_1 \cdot v_2 = 0$.

2. Transformations Answer yes/no and give short explanation.

   (a) (6 points) Does non-uniform scaling in 2D preserve angles?

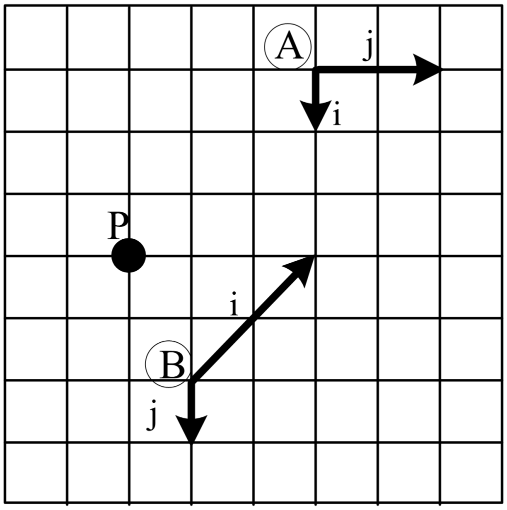
   No. Consider the angle θ that the vector $(1, 1)$ makes with the x axis. Scaling y by 2 creates the vector $(1, 2)$. The angle ϕ that this vector makes with the x axis is different. $\theta \neq \phi$, the angle is not preserved.

   (b) (6 points) Does shear in 2D (e.g. below) preserve parallel lines?

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

   Yes. A shear transformation is an affine transformation. All affine transformations preserve lines and parallelism.

3. Transformation as a Change of Coordinate Frame



(a) (7 points) Specify the coordinates of point P with respect to coordinate frames A and B.

$$P_A = \begin{pmatrix} 3 \\ -1\frac{1}{2} \end{pmatrix} P_B = \begin{pmatrix} -\frac{1}{2} \\ -3 \end{pmatrix}$$

(b) (8 points) Derive a transformation that takes a point from frame A to frame B, i.e., determine $M_{A\rightarrow B}$, where $P_B = P_A M_{A\rightarrow B}$. Verify your solution using your answer to part (a).

We find

$$M_{A\rightarrow B} = \begin{pmatrix} a & c & e \\ b & d & f \\ 0 & 0 & 1 \end{pmatrix}$$

Where

$$A_i = aB_i + bB_j$$

$$A_j = cB_i + dB_j$$

$$A_O - B_O = eB_i + fB_j$$

A_O denotes the origin of A and B_O denotes the origin of B. This matrix is

$$M_{A\rightarrow B} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

4. (a) (8 points) Decompose the following complex transformations in homogeneous coordinates into a product of simple transformations (scaling, rotation, translation, shear). Pay attention to the order of transformations.

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is a product $T * S * R$ or $S * T * R$ of the translation T, the scaling S, and rotation R where

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} S = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) (7 points) What is the inverse of this transformation matrix (3D homogenous coordinates)? Explain your computation.

The inverse is a product of the inverse transformations R^{-1}, T^{-1}, and S^{-1} applied in reverse order $R^{-1} * S^{-1} * T^{-1}$ or $R^{-1} * T^{-1} * S^{-1}$. So

$$R^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} S^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} T^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And the answer is

$$R^{-1} * T^{-1} * S^{-1} = R^{-1} * S^{-1} * T^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(c) (7 points) Give the sequence of OpenGL transformations that would produce the same transformation matrix as in (a) above.

$glTranslatef(0.0, 0.1, 0.0);$
$glScalef(2.0, 0.0, 0.0);$
$glRotatef(-90, 1.0, 0.0, 0.0);$

5. (a) (7 points) Given 3 points $P_1 = (0,0,0)$, $P_2 = (2,0,2)$ and $P_3 = (0,2,0)$ in 3D compute the implicit equation $Ax + By + +Cz + D = 0$ of the plane passing through those points. Show your work.

   First we compute the normal to plane N.

$$N = \frac{(P_2 - P_1) \times (P_3 - P_1)}{\|(P_2 - P_1) \times (P_3 - P_1)\|}$$

   We know that
   $A = N_x = -\frac{1}{\sqrt{2}}$
   $B = N_y = 0$
   $C = N_z = \frac{1}{\sqrt{2}}$

   Since the plane instersects the origin at P_1 we know that $D = 0$. The plane equation is
   $-\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}z = 0$
   This can be simplified to
   $z - x = 0$

   (b) (7 points) Does the point $P = (-1, 1, -1)$ lie on the plane you just computed? Explain your computation.

   Yes. Simply plug P into the plane equation. If the result is 0 (which it is) then it is on the plane.

   (c) (10 points) Write an algorithm (**in 3D**) that tests is a point is inside a triangle P_1, P_2, P_3. Explain your algorithm. Are there special cases the algorithm must handle?

   Here are a couple options. First, try to express the test point P in barycentric coordinates. These are the coordinates (a_1, a_2, a_3) such that $P = a_1 * P_1 + a_2 * P_2 + a_3 * P_3$. When these coordinates satisfy the inequality $0 \leq a_1, a_2, a_3 \leq 1$ and the equality $a_1 + a_2 + a_3 = 1$ then the point lies in the triangle. We find the coordinates by solving the following equations.
   $a_1 = A_{P_2 P_3 P}/A$
   $a_2 = A_{P_3 P_1 P}/A$
   $a_3 = A_{P_1 P_2 P}/A$
   Where A is the area of the triangle P_1, P_2, P_3 and $A_{P_x P_y P_z}$ is the area of triangle of the points P_x, P_y, P_z.

   Another way is rotate the points so they lie parallel to the xy plane. We must make the triangle normal and the z axis collinear. First, derive the plane equation for the points (see above) to get N. Next, derive the axis of rotation as $N \times Z$ where Z is the z axis. The angle of rotation is $cos^{-1}(N_z)$. Apply this rotation to all the points. If the z of P is different from P_1, P_2, P_3 then we reject P because it lies in a different plane than the triangle. Finally, we ignore the points' z components and derive the implicit line equations for the line segments of the triangle, plug in the coordinates of P into each equation and check if the signs are all the same. If they are, then P lies inside the triangle.

6. (15 points) Write the 2D transformation that mirrors (reflects) space around the line $x - y = 0$.

Reflections across this line $y = x$ swaps the x and y coordinates. This is easily expressed by the following matrix.

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

You can think of swapping coordinates as simply swapping rows in the identity matrix.