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Chapter 4

Transformations

Transformations

- Transformation = *one-to-one* and *onto* mapping of R^n to itself
- *Affine* transformation – $T(\mathrm{v}) = \mathrm{Av+b}$
  - A – matrix
  - v, b – vectors

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

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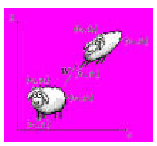
Geometric Transformations

- *Geometric Transformation* = affine transformation with geometric meaning



- Mathematically transformations are defined on vectors $\Rightarrow$ for point P, use vector P-Origin

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Matrix Representation

- Combining transformations

$$ST(v) = C(Av+b)+d = (CA)v + (Cb+d)$$

- Same format (multiply by matrix & add vector)

$$A' = CA \qquad b' = Cb + d$$

- Can reduce to matrix multiplication only
  - Homogeneous coordinates (later on)

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Transformations – sub topics

- 2D Transformations
- Homogeneous Coordinates
- 3D Transformations
- Composing Transformations
- Transformation Hierarchies
- Transforming Normals
- Assignment 1 – dinosaurs
  - Use transformations to create and animate dinosaurs made from spheres

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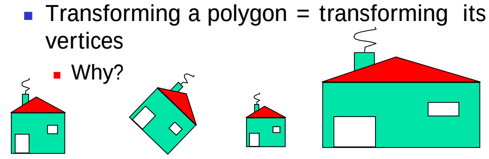
Transformations

- Transforming an object = transforming all its points
- Transforming a polygon = transforming its vertices
  - Why?



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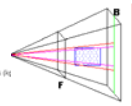
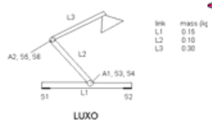
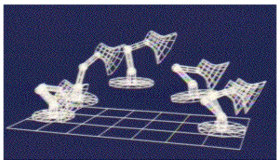
Applications

- Viewing (more later)
- Modeling
- Articulation







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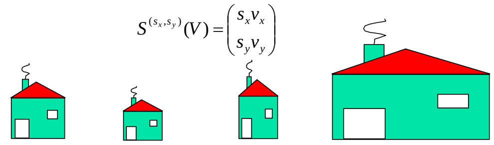
Scaling

- $V = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ – vector in XY plane
- *Scaling* operator S with parameters (s_x, s_y):

$$S^{(s_x,s_y)}(V) = \begin{pmatrix} s_x v_x \\ s_y v_y \end{pmatrix}$$



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Scaling

- Matrix form:

$$S^{(s_x,s_y)}(V) = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} s_x v_x \\ s_y v_y \end{pmatrix}$$

- Independent in x and y

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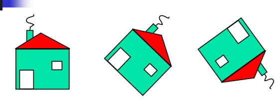
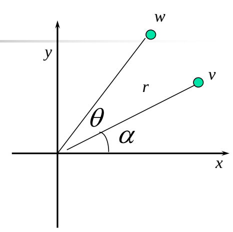
Rotation





- Polar form:

$$v = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix}$$

- Rotating v counterclockwise by θ to w:

$$w = \begin{pmatrix} w_x \\ w_y \end{pmatrix} = \begin{pmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{pmatrix} = \begin{pmatrix} r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \\ r\cos\alpha\sin\theta + r\sin\alpha\cos\theta \end{pmatrix}$$

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Rotation

- Matrix form:

$$w = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} v$$

- *Rotation* operator R (at the origin) with parameter θ:

$$R^{\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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Rotation Properties

- R^{θ} is orthonormal

$$\left(R^{\theta}\right)^{-1} = \left(R^{\theta}\right)^{T}$$

- $R^{-\theta}$ - rotation by $-\theta$ is

$$R^{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \left(R^{\theta}\right)^{-1}$$

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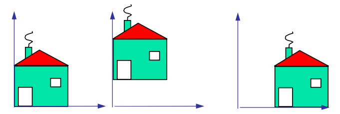
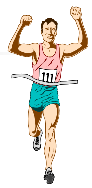
Translation





- Translation operator *T* with parameters (t_x, t_y):

$$T^{(t_x,t_y)}(v) = \begin{pmatrix} v_x + t_x \\ v_y + t_y \end{pmatrix}$$

- How can we write this in matrix form?

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Translation - Homogeneous Coordinates

- To represent T in matrix form – introduce homogeneous coordinates:

$$v = \begin{pmatrix} v_x \\ v_y \end{pmatrix} \rightarrow v^h = \begin{pmatrix} v_x^h \\ v_y^h \\ v_w^h \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}$$

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Translation - Homogeneous Coordinates

- Conversion (projection) from homogeneous space to Euclidean:

$$v = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_x^h / v_w^h \\ v_y^h / v_w^h \end{pmatrix}$$

- Projections is not 1:1

$$\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0.5 \end{pmatrix} \text{ all project to } \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

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Translation

- Using homogeneous coordinates, translation operator may be expressed as:

$$T^{(t_x,t_y)}(v^h) = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix} = \begin{pmatrix} v_x + t_x \\ v_y + t_y \\ 1 \end{pmatrix}$$

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Homogeneous Coordinates

$$\textbf{R}otation = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\textbf{S}cale = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Other ideas for uniform scale?

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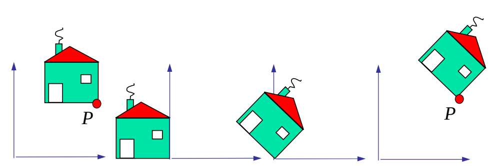
Transformation Composition

- What operation rotates XY by θ around $P = \begin{pmatrix} p_x \\ p_y \end{pmatrix}$?
- Answer:
  - Translate *P* to origin
  - Rotate around origin by θ
  - Translate back



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Transformation Composition

$$T^{(p_x,p_y)}R^{\theta}T^{(-p_x,-p_y)}(V)$$
$$=\begin{bmatrix}1&0&p_x\\0&1&p_y\\0&0&1\end{bmatrix}\begin{bmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-p_x\\0&1&-p_y\\0&0&1\end{bmatrix}\begin{pmatrix}v_x\\v_y\\1\end{pmatrix}$$

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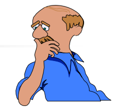
Transformations Quiz



- What do these transformations do?

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}-1&0\\0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0\\1&1\end{bmatrix}$$

$$\begin{bmatrix}1&a\\0&1\end{bmatrix}$$

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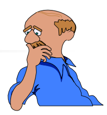
Transformations Quiz



- And these homogeneous ones?

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$$
$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0.5\end{bmatrix}$$

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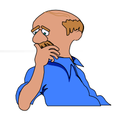
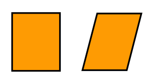
Transformations Quiz



- How to mirror through arbitrary line in XY?
- What transformation achieves this?



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Shear & Mirroring/Reflection

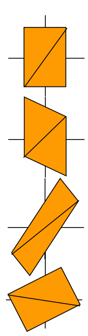
- Shear (canonical)

$$\begin{bmatrix}1&0\\1&1\end{bmatrix},\begin{bmatrix}1&1\\0&1\end{bmatrix},\begin{bmatrix}1&a\\0&1\end{bmatrix},\begin{bmatrix}1&0\\a&1\end{bmatrix}$$

- Mirroring/Reflection

$$\begin{bmatrix}1&0\\0&-1\end{bmatrix},\begin{bmatrix}-1&0\\0&1\end{bmatrix}$$

- What is the relation between shears and rotations?



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Linear Transformations

- Combinations of
  - shear
  - scale
  - rotate
  - reflect

$$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\qquad \begin{aligned}x'&=ax+by\\y'&=cx+dy\end{aligned}$$

- Properties (why?)
  - satisfies T(sx+ty) = s T(x) + t T(y)
  - origin maps to origin
  - Straight lines map to straight lines
  - parallel lines remain parallel
  - closed under composition

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Affine Transformations

- Combinations of
  - linear transformations
  - translations

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties (why?)
  - origin does not necessarily map to origin
  - lines map to lines
  - parallel lines remain parallel
  - ratios are preserved
  - closed under composition

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3D Transformations

- All 2D transformations extend to 3D
- In homogeneous coordinates:

Scaling

$$S^{(s_x,s_y,s_z)} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation

$$T^{(t_x,t_y,t_z)} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation around the z axis

$$R_z^\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

glScalef(a,b,c); **glTranslatef(a,b,c);** **glRotatef(angle,0,0,1);**

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3D Rotation in X, Y

around x axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

glRotatef(angle,1,0,0);

around y axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

glRotatef(angle,0,1,0);

- general OpenGL command

glRotatef(angle,x,y,z);

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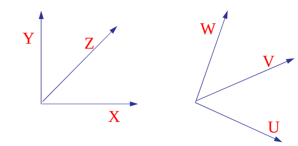
Arbitrary Rotation



- Problem:
  - Given two orthonormal coordinate systems *XYZ* and *UVW*
  - Find transformation from one to the other
- Answer:
  - Transformation matrix R whose columns are *U,V,W:*

$$R = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix}$$

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Arbitrary Rotation

- Proof:

$$R(X) = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = U$$

- Similarly $R(Y) = V$ & $R(Z) = W$

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Arbitrary Rotation (cont.)

- Inverse (=transpose) transformation R^{-1} provides mapping from *UVW* to *XYZ*
- E.g.

$$R^{-1}(U) = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} u_x^2 + u_y^2 + u_z^2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = X$$

- Comment: Used for mapping between XY and arbitrary plane

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3D Shear

- shear in x

$$xshear(sy,sz)=\begin{bmatrix}1 & sy & sz & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

- shear in y

$$yshear(sx,sz)=\begin{bmatrix}1 & 0 & 0 & 0\\sx & 1 & sz & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

- shear in z

$$zshear(sx,sy)=\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\sx & sy & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

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Undoing Transformations: Inverses

$$\mathbf{T}(x,y,z)^{-1} = \mathbf{T}(-x,-y,-z)$$
$$\mathbf{T}(x,y,z)\ \mathbf{T}(-x,-y,-z) = \mathbf{I}$$

$$\mathbf{R}(z,\theta)^{-1} = \mathbf{R}(z,-\theta) = \mathbf{R}^{T}(z,\theta)$$ **(R is orthogonal)**
$$\mathbf{R}(z,\theta)\ \mathbf{R}(z,-\theta) = \mathbf{I}$$

$$\mathbf{S}(sx,sy,sz)^{-1} = \mathbf{S}(\frac{1}{sx},\frac{1}{sy},\frac{1}{sz})$$
$$\mathbf{S}(sx,sy,sz)\,\mathbf{S}(\frac{1}{sx},\frac{1}{sy},\frac{1}{sz}) = \mathbf{I}$$

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3D Transformations - Composition

- Questions:
  - Is $S_1S_2 = S_2S_1$?
  - Is $T_1T_2 = T_2T_1$?
  - Is $R_1R_2 = R_2R_1$?
  - Is $S_1R_2 = R_2S_1$?
  - .....

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Composing Translations

$$T1 = T(dx_1,dy_1) = \begin{bmatrix}1 & & & dx_1\\ & 1 & & dy_1\\ & & 1 & \\ & & & 1\end{bmatrix} \qquad T2 = T(dx_2,dy_2) = \begin{bmatrix}1 & & & dx_2\\ & 1 & & dy_2\\ & & 1 & \\ & & & 1\end{bmatrix}$$

$$P'' = T2 \bullet P' = T2 \bullet [T1 \bullet P] = [T2 \bullet T1] \bullet P, where$$

$$T2 \bullet T1 = \begin{bmatrix}1 & & & dx_1 + dx_2\\ & 1 & & dy_1 + dy_2\\ & & 1 & \\ & & & 1\end{bmatrix}$$

Translations add

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Composing Transformations

- scaling

$$S2 \bullet S1 = \begin{bmatrix}sx_1 \bullet dx_2 & & & \\ & sy_1 \bullet sy_2 & & \\ & & 1 & \\ & & & 1\end{bmatrix}$$

scales multiply

- rotation

$$R2 \bullet R1 = \begin{bmatrix}\cos(\theta 1+\theta 2) & -\sin(\theta 1+\theta 2) & & \\ \sin(\theta 1+\theta 2) & \cos(\theta 1+\theta 2) & & \\ & & 1 & \\ & & & 1\end{bmatrix}$$

rotations add

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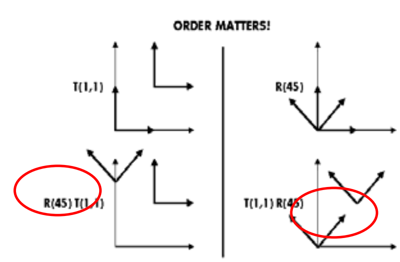
Composing Transformations



Ta Tb = Tb Ta, but Ra Rb != Rb Ra and Ta Rb != Rb Ta

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Another Transformations Quiz

- What does each transformation preserve?

	lines	parallel lines	distance	angles	normals	convexity
scaling						
rotation						
translation						
shear						

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