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Chapter 2

Math Review

3D (or 2D) space - descriptors

- Describe position in space
  - Vectors
- Describe transformation
  - Operations on vectors
  - Matrices (Transformations)

Vectors

- Notations
  - column vectors

$$a_{col} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix}$$

$$a_{col}^T = a_{row}$$

  - row vectors

$$a_{row} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$$

- Simple operations:
  - sum (subtract), multiply by scalar

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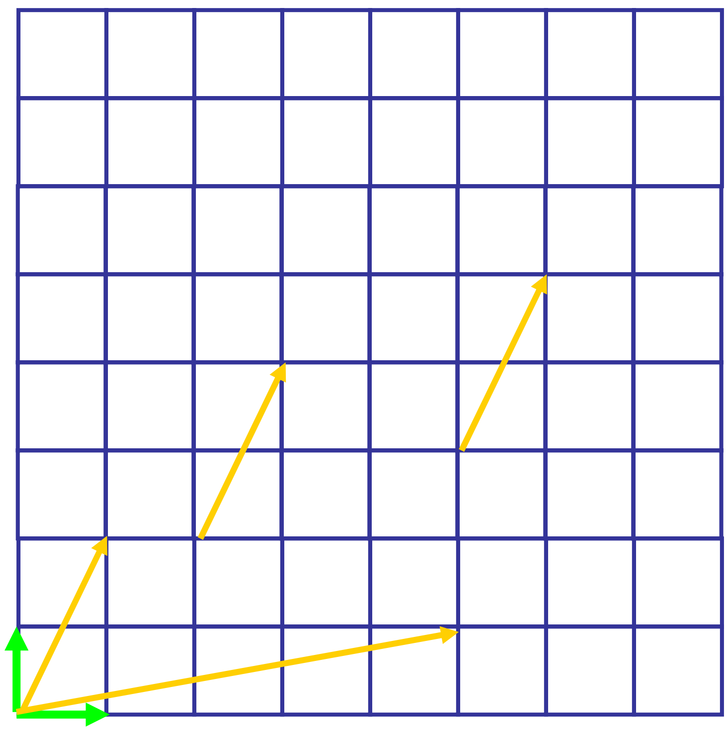
Vectors

- Geometric meaning – oriented segment in nD space
- Location (point) if given origin – our setting



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Dot Product

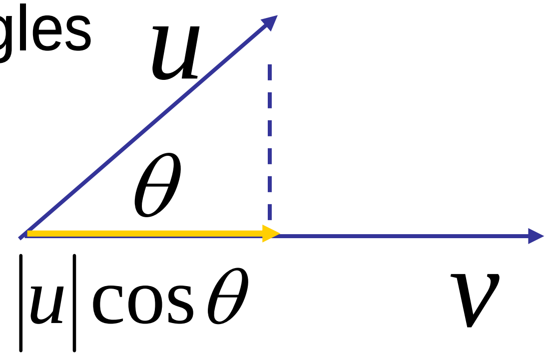
- Or *inner product*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \bullet \begin{bmatrix} a \\ b \\ c \end{bmatrix} = x*a + y*b + z*c$$

$$P \bullet N$$

- Geometric interpretation
  - Useful for computing angles

$$u \bullet v = \|u\| \|v\| \cos\theta$$



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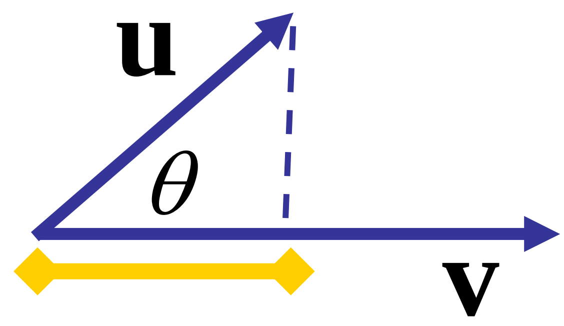
Dot Product Geometry

- length of projection of u onto v

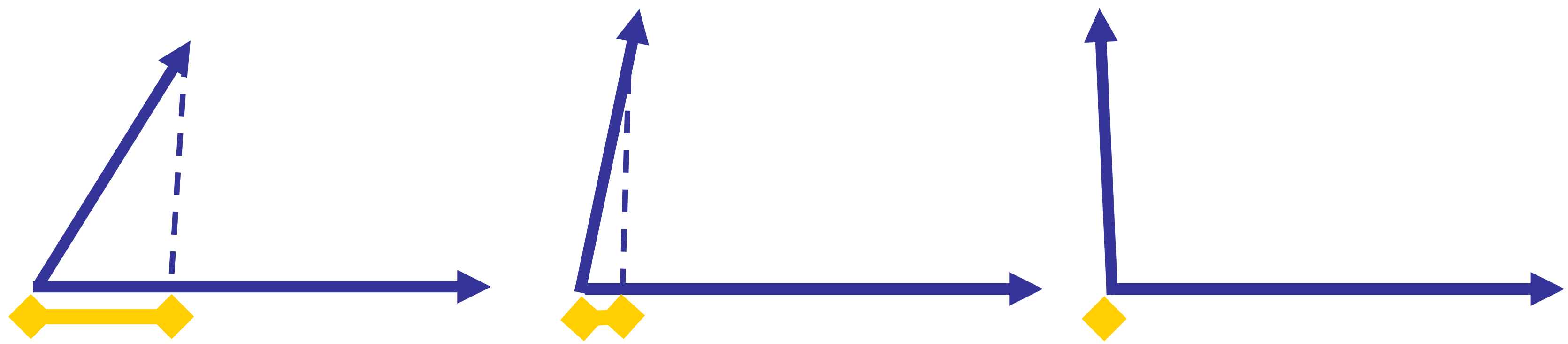


$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

$$\|\mathbf{u}\|\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|}$$

$\|\mathbf{u}\|\cos\theta$

- as lines become perpendicular, $\mathbf{u} \bullet \mathbf{v} \to 0$



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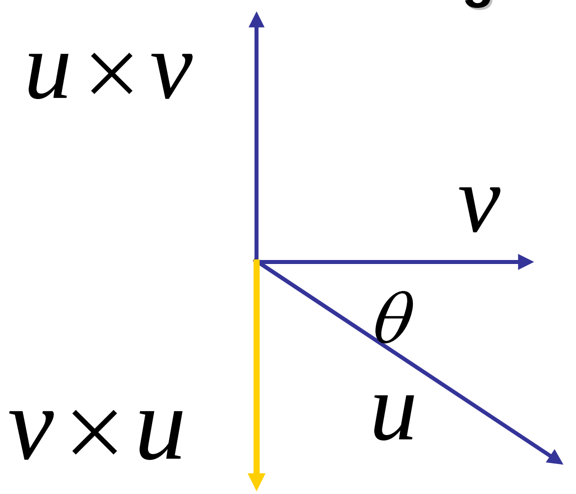
Cross Product

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Right Handed Coordinate System



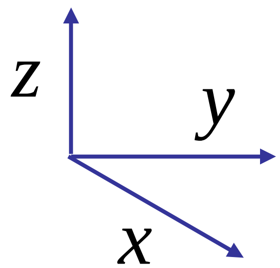
(curl fingers from u to v; thumb points to u x v)

$$\|u \times v\| = \|u\|\|v\| \sin\theta$$

Math Review

- 3D Coordinate Systems

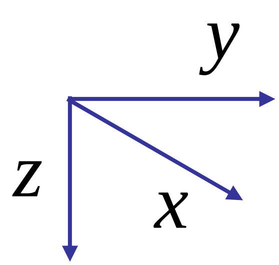
Right-handed Coordinate System



$$z = x \times y$$

using right-hand rule

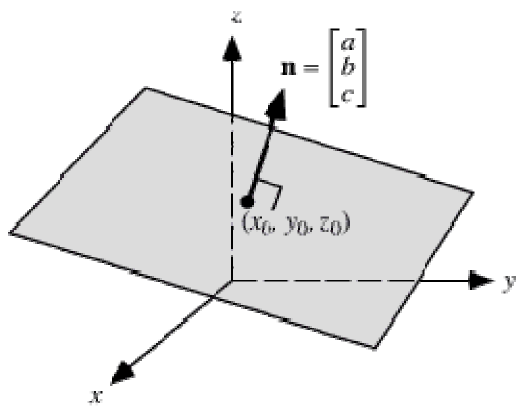
Left-handed Coordinate System

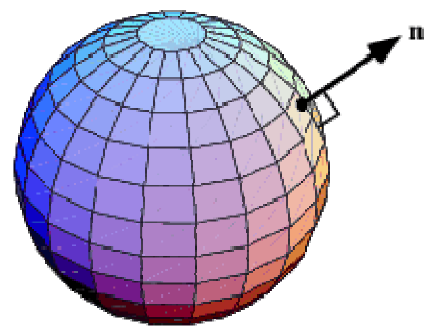


$$z = x \times y$$

using left-hand rule

Normal

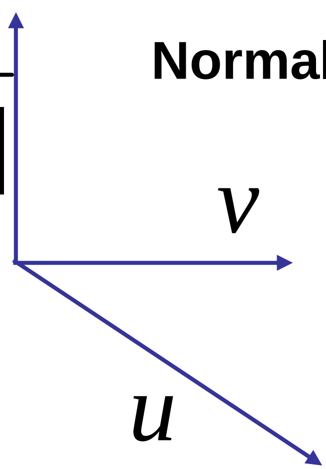




Normal: unit vector perpendicular to surface
Unit vector – vector of length 1

$$\frac{u \times v}{\| u \times v \|}$$

Normal to plane containing u **&** v



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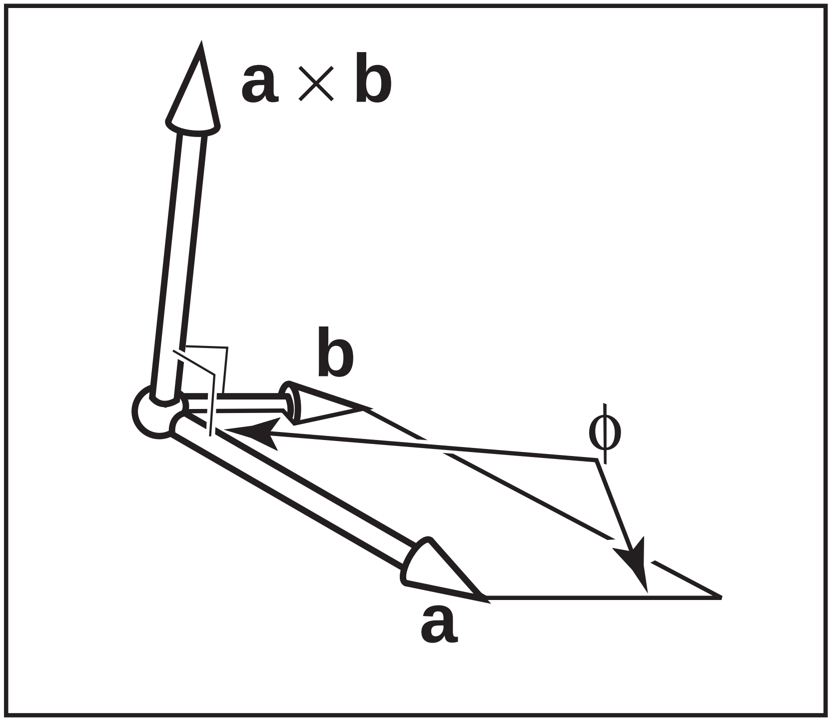
Cross-Product – geometric meaning

- Direction = normal to plane defined by vectors

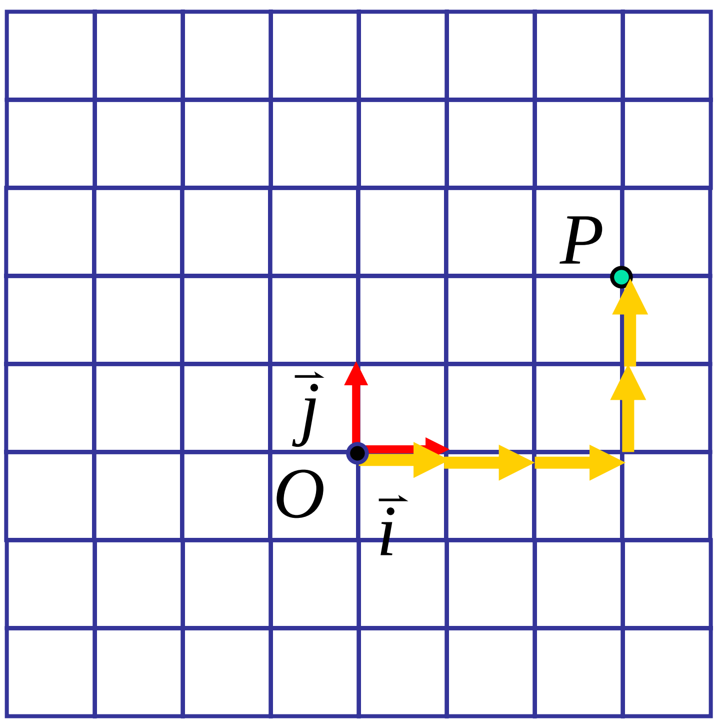
- Size = parallelogram area

$$\|a \times b\| = \|a\|\|b\| \sin\theta$$



Coordinate Frame

- Coordinate frame: basis (independent) vectors + origin
  - can specify location - *points*



$$P = O + x\vec{i} + y\vec{j}$$

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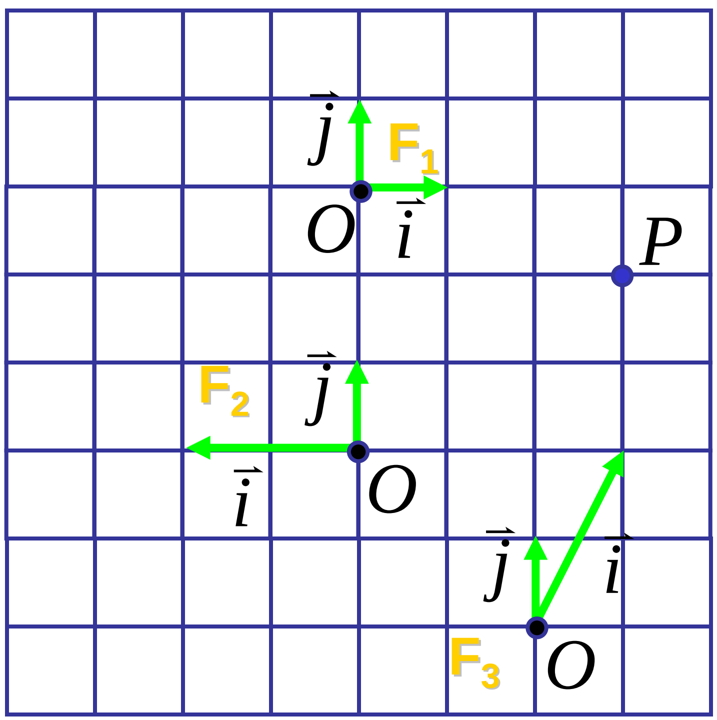
Math Review

- Working with Frames



$$P = O + x\vec{i} + y\vec{j}$$

F_1 **P(3,-1)**

F_2 **P(-1.5,2)**

F_3 **P(1,2)**

Matrices

- Matrices in CG – tools to manipulate vectors
- Addition/Subtraction

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} n_{11}+m_{11} & n_{12}+m_{12} \\ n_{21}+m_{21} & n_{22}+m_{22} \end{bmatrix}$$

- example

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 5 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 1+(-2) & 3+5 \\ 2+7 & 4+1 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 9 & 5 \end{bmatrix}$$

Scalar-Matrix Multiplication

- scalar * matrix = matrix

$$a\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} a * m_{11} & a * m_{12} \\ a * m_{21} & a * m_{22} \end{bmatrix}$$

- example

$$3\begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3*2 & 3*4 \\ 3*1 & 3*5 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 3 & 15 \end{bmatrix}$$

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Matrix-Matrix Multiplication

- row by column

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$p_{11} = m_{11}n_{11} + m_{12}n_{21}$$

Matrix-Matrix Multiplication

- row by column

$$\begin{bmatrix} m_{11} & m_{12} \\ \boxed{m_{21} \quad m_{22}} \end{bmatrix} \begin{bmatrix} \boxed{n_{11}} & n_{12} \\ \boxed{n_{21}} & n_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ \boxed{p_{21}} & p_{22} \end{bmatrix}$$

$$p_{11} = m_{11}n_{11} + m_{12}n_{21}$$

$$p_{21} = m_{21}n_{11} + m_{22}n_{21}$$

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Matrix-Matrix Multiplication

- row by column

$$\begin{bmatrix} \boxed{m_{11} \quad m_{12}} \\ m_{21} \quad m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & \boxed{\begin{matrix} n_{12} \\ n_{22} \end{matrix}} \\ n_{21} & \end{bmatrix} = \begin{bmatrix} p_{11} & \boxed{p_{12}} \\ p_{21} & p_{22} \end{bmatrix}$$

$$p_{11} = m_{11}n_{11} + m_{12}n_{21}$$

$$p_{21} = m_{21}n_{11} + m_{22}n_{21}$$

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$$p_{11} = m_{11}n_{11} + m_{12}n_{21}$$

$$p_{21} = m_{21}n_{11} + m_{22}n_{21}$$

$$p_{12} = m_{11}n_{12} + m_{12}n_{22}$$

$$p_{22} = m_{21}n_{12} + m_{22}n_{22}$$

- noncommutative: **AB != BA**

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Matrix Multiplication

- can only multiply if
number of left cols = number of right rows
  - legal

$$\begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix} \begin{bmatrix} h & i & j \\ k & l & m \\ n & o & p \end{bmatrix}$$

  - undefined

$$\begin{bmatrix} a & b & c \\ e & f & g \\ o & p & q \end{bmatrix} \begin{bmatrix} h & i & j \\ k & l & m \end{bmatrix}$$

Matrix-Vector Multiplication

- points as column vectors: postmultiply

$$\begin{bmatrix} x' \\ y' \\ z' \\ h' \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ h \end{bmatrix} \qquad \mathbf{p}' = \mathbf{M}\mathbf{p}$$

- points as row vectors: premultiply

$$\begin{bmatrix} x' & y' & z' & h' \end{bmatrix} = \begin{bmatrix} x & y & z & h \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}^T \qquad \mathbf{p}'^T = \mathbf{p}^T \mathbf{M}^T$$

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Matrices

- transpose

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}^T = \begin{bmatrix} m_{11} & m_{21} & m_{31} & m_{41} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix}$$

- identity

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- inverse

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

  - not all matrices are invertible

Matrices and Linear Systems

- linear system of n equations, n unknowns

$$3x + 7y + 2z = 4$$
$$2x - 4y - 3z = -1$$
$$5x + 2y + z = 1$$

- matrix form **Ax**=**b**

$$\begin{bmatrix} 3 & 7 & 2 \\ 2 & -4 & -3 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

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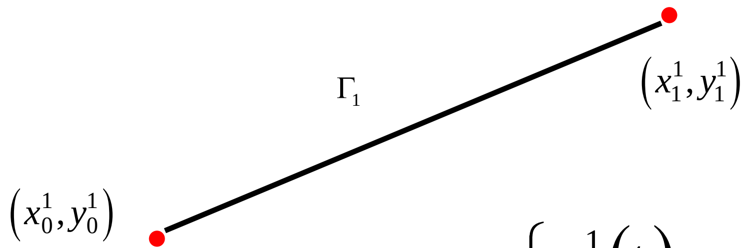
Lines & Segments (2D)

- Segment Γ_1 from $P_0 = (x_0^1, y_0^1)$ to $P_1 = (x_1^1, y_1^1)$



$$G_1 = \begin{cases} x^1(t) = x_0^1 + (x_1^1 - x_0^1)t \\ y^1(t) = y_0^1 + (y_1^1 - y_0^1)t \end{cases} t \in [0,1]$$

- Line through $P_0 = (x_0^1, y_0^1)$ and $P_1 = (x_1^1, y_1^1)$
  - Parametric $G_1(t), t \in (-\infty, -\infty)$
  - Implicit $Ax+By+C=0$
    - Solve 2eq in 2 unknowns (set $A^2+B^2=1$)
  - 3D?

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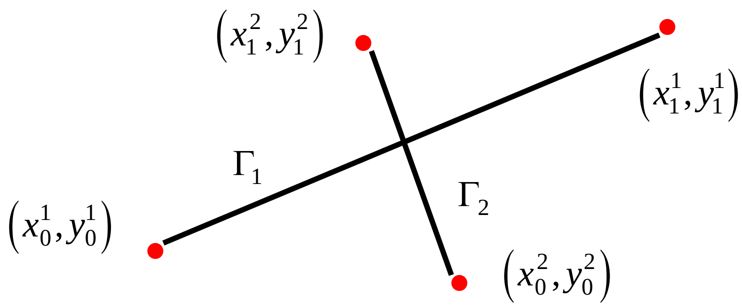
Line-Line Intersection



$$G_1 = \begin{cases} x^1(t) = x_0^1 + \left(x_1^1 - x_0^1\right)t \\ y^1(t) = y_0^1 + \left(y_1^1 - y_0^1\right)t \end{cases} \quad t \in [0,1] \qquad G_2 = \begin{cases} x^2(r) = x_0^2 + \left(x_1^2 - x_0^2\right)r \\ y^2(r) = y_0^2 + \left(y_1^2 - y_0^2\right)r \end{cases} \quad r \in [0,1]$$

Intersection: *x* & *y* values equal in both representations - two linear equations in two unknowns (*r*,*t*)

$$\begin{aligned} x_0^1 + \left(x_1^1 - x_0^1\right)t &= x_0^2 + \left(x_1^2 - x_0^2\right)r \\ y_0^1 + \left(y_1^1 - y_0^1\right)t &= y_0^2 + \left(y_1^2 - y_0^2\right)r \end{aligned}$$

Question: What is the meaning of $r,t < 0$ or $r,t > 1$?

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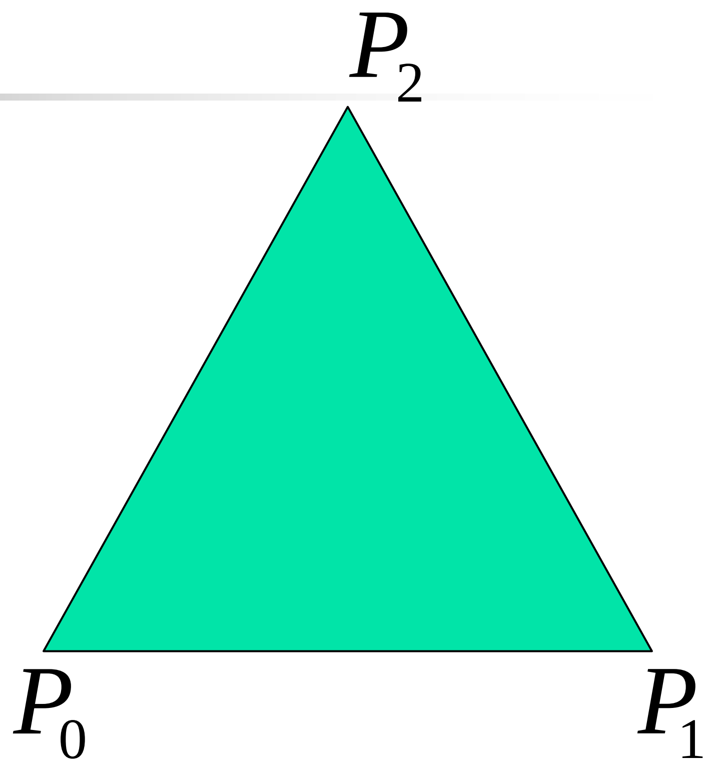
Triangle

- Normal

$$n = \frac{(P_1 - P_0)\times(P_2 - P_0)}{\left\|(P_1 - P_0)\times(P_2 - P_0)\right\|}$$

- Area

$$A = \frac{1}{2}\left\|\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}\right\|$$



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Plane

- Implicit equation: $Ax+By+Cz+D=0$
  - Normalize (one option): $A^2+B^2+C^2=1$
  - (A,B,C) - normal to plane
- To find given 3 points $P_0 \quad P_1 \quad P_2$ in the plane:

$$n = \frac{(P_1 - P_0) \times (P_2 - P_0)}{\left\| (P_1 - P_0) \times (P_2 - P_0) \right\|}$$

- Get $n_x x + n_y y + n_z z + D = 0$ (solve 1 eq to get D)