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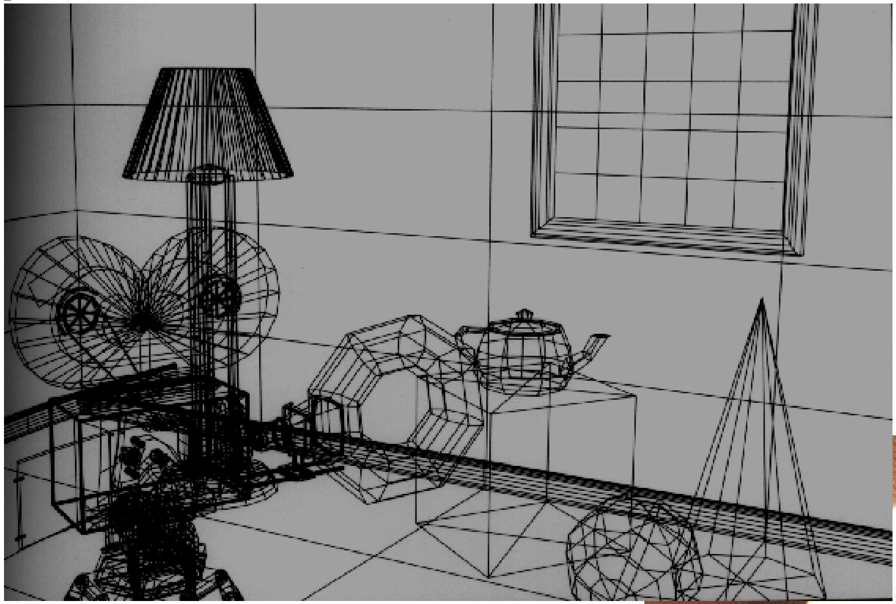
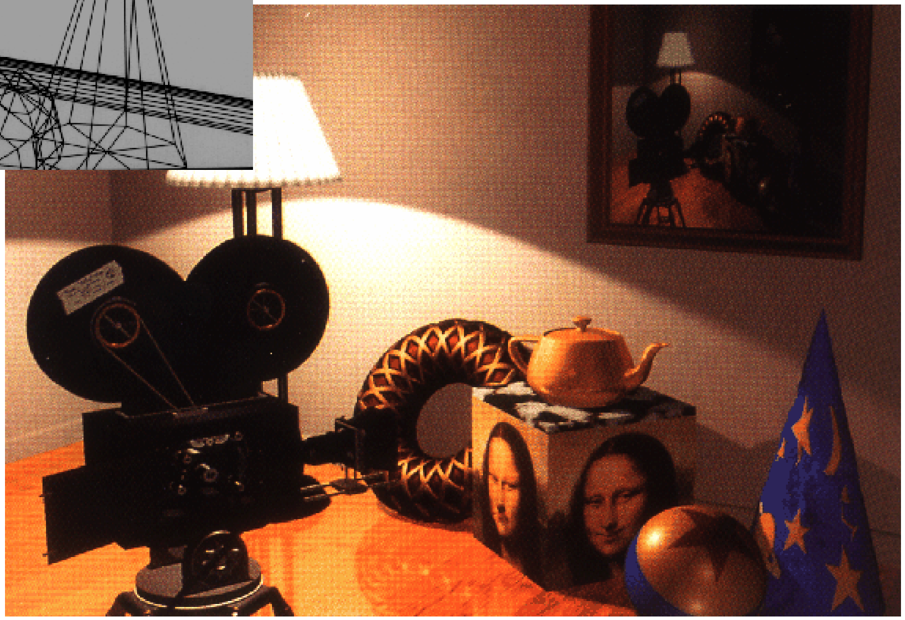
Chapter 14

Geometric Modeling Basics
Terminology & Splines

An Example





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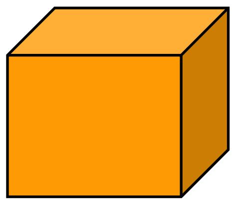
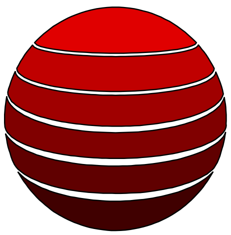
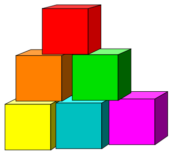
Geometry

- Mathematical models of real world objects shape
  - Boundary representations
    - Freeform
    - Mesh
  - Volumetric representations
    - Primitive based
    - Voxels





- We will
  - Talk about boundary reps
  - Focus on curves



Freeform Representation - Curves

- Explicit form: y=y(x)
- Implicit form: f(x,y)=0
- Parametric form: [x(t),y(t)]
- Example – origin centered circle of radius *R:*

Explicit is a special case of implicit and parametric form

Explicit :

$$y = +\sqrt{R^2 - x^2} \cup y = -\sqrt{R^2 - x^2}$$

Implicit :

$$x^2 + y^2 - R^2 = 0$$

Parametric :

$$(x, y) = (R\cos\theta, R\sin\theta), \theta \in [0, 2\pi]$$

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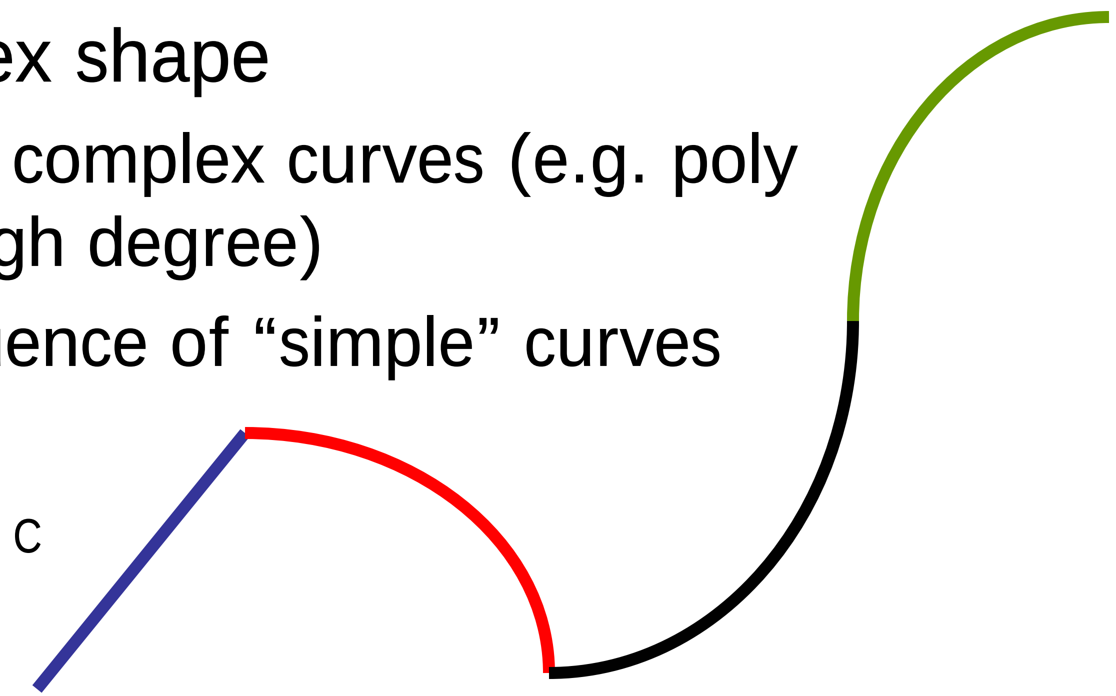
Parametric Curves

- Describe geometry (2D)
- Describe path (2D/3D)
- Typically parametric
  $C(t)=[x(t),y(t)]$
- Complex shape
  - Very complex curves (e.g. poly of high degree)
  - Sequence of "simple" curves



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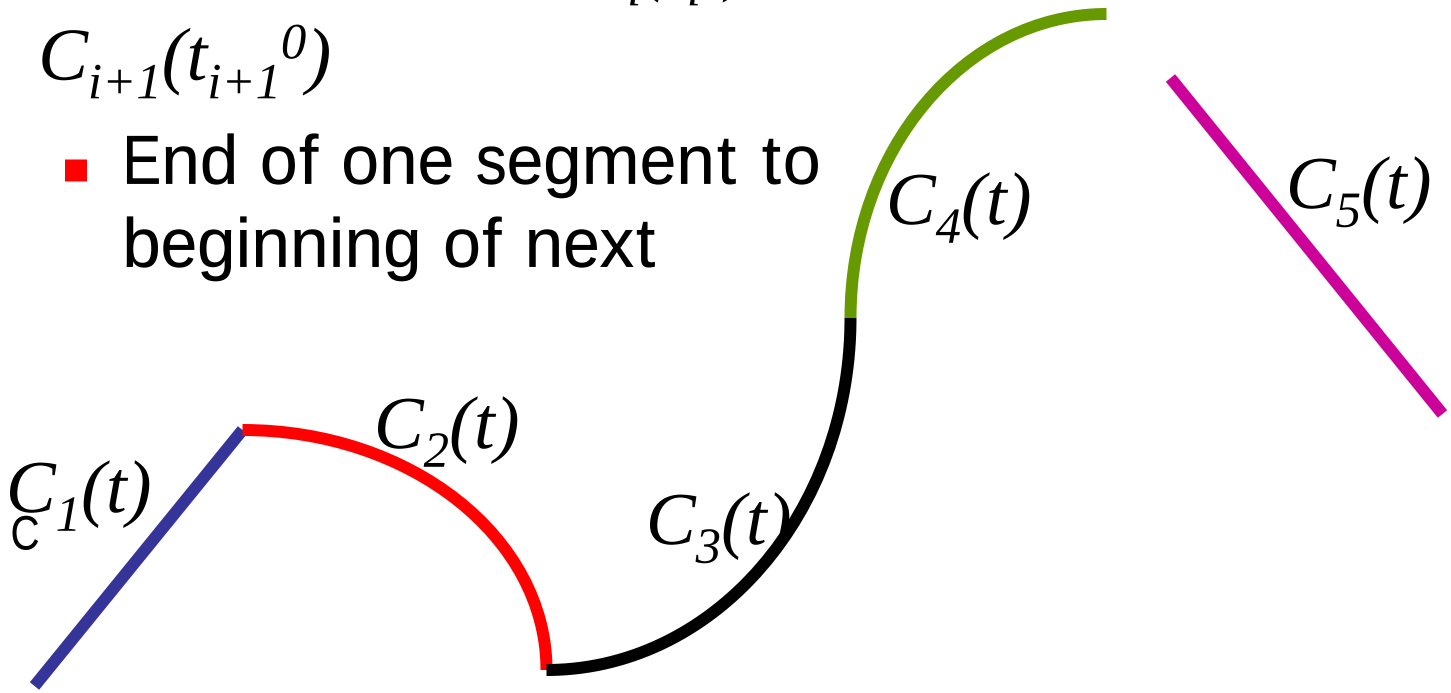
Sequence of Curves

- $C_1(t), C_2(t),.., C_n(t)$
- Each curve $C_i(t)$ has parameter domain $t \in [t_i^0, t_i^1]$
- How to connect $C_i(t_i^1)$ to $C_{i+1}(t_{i+1}^0)$
  - End of one segment to beginning of next



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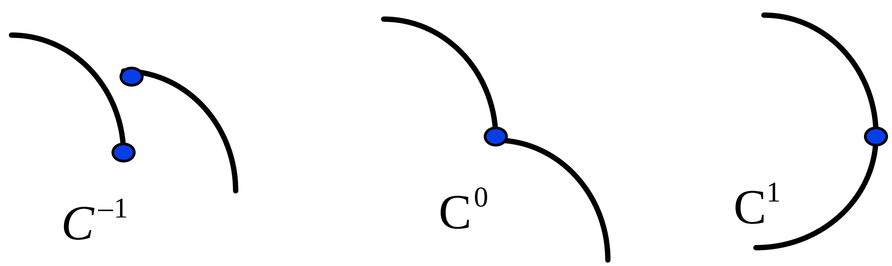
Continuity

- $C_1(t)$ & $C_2(t)$, $t \in [0,1]$ - parametric curves
- Level of continuity at $C_1(1)$ and $C_2(0)$ is:
  - C^{-1}: $C_1(1) \neq C_2(0)$ (discontinuous)
  - C^0: $C_1(1) = C_2(0)$ (positional continuity)
  - C^k, $k > 0$: continuous up to k-th derivative



$$C_1^{(j)}(1) = C_2^{(j)}(0), 0 \le j \le k$$

- Continuity of single curve defined similarly

Geometric Continuity

- Analytic continuity - too strong a requirement
- Geometric continuity – common curve is geometrically smooth (per given level *k*)
  - $G^k, k \leq 0$: Same as C^k
  - $G^k\ k = 1: C'_1(1) = \alpha C'_2(0)\ \alpha > 0$
  - $G^k\ k \geq 0$: In arc-length reparameterization of $C_1(t)$ & $C_2(t)$, the two are C^k

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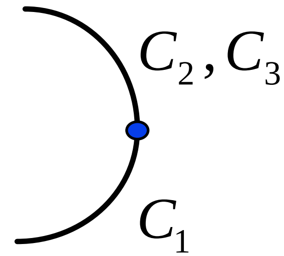
Geometric Continuity

- E.g.

$$C_1(t) = [\cos(t), \sin(t)] \quad t \in [-0.5\pi, 0]$$
$$C_2(t) = [\cos(t), \sin(t)] \quad t \in [0, 0.5\pi]$$
$$C_3(t) = [\cos(2t), \sin(2t)] \quad t \in [0, 0.25\pi]$$



  - $C_1(t)$ & $C_2(t)$ are C^k (& G^k) continuous
  - $C_1(t)$ & $C_3(t)$, are G^k continuous (not C^k)

Splines – Free Form Curves

- Usually parametric
  - $C(t)=[x(t),y(t)]$ or $C(t)=[x(t),y(t),z(t)]$
- Description = basis functions + coefficients

$$C(t) = \sum_{i=0}^{n} P_i B_i(t) = (x(t), y(t))$$

$$x(t) = \sum_{i=0}^{n} P_i^x B_i(t)$$

$$y(t) = \sum_{i=0}^{n} P_i^y B_i(t)$$

  - Same basis functions for all coordinates

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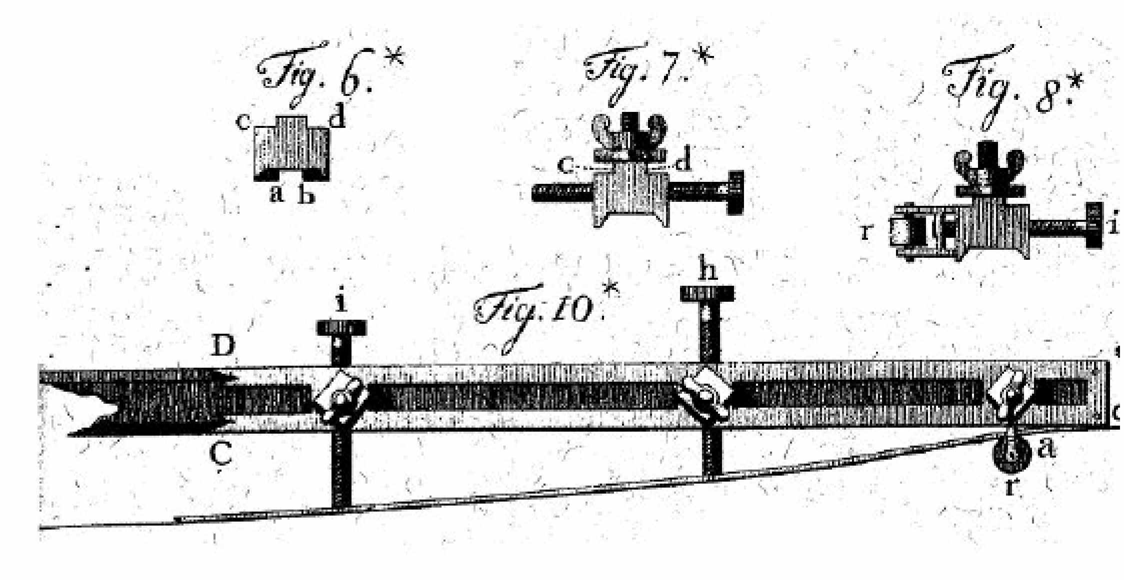
Splines – Free Form Curves

- Geometric meaning of coefficients (base)
  - Approximate/interpolate set of positions, derivatives, etc..



- Will see one example

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Hermite Cubic Basis

- Geometrically-oriented coefficients
  - 2 positions + 2 tangents
- Require $C(0)=P_0$, $C(1) = P_1$, $C'(0)=T_0$, $C'(1)=T_1$
- Define basis function per requirement

$$C(t) = P_0 h_{00}(t) + P_1 h_{01}(t) + T_0 h_{10}(t) + T_1 h_{11}(t)$$

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Hermite Basis Functions

$$C(t) = P_0 h_{00}(t) + P_1 h_{01}(t) + T_0 h_{10}(t) + T_1 h_{11}(t)$$

- To enforce $C(0)=P_0$, $C(1) = P_1$, $C'(0)=T_0$, $C'(1)=T_1$ basis should satisfy

$$h_{ij}(t): i, j = 0,1, t \in [0,1]$$

curve	$C(0)$	$C(1)$	$C'(0)$	$C'(1)$
$h_{00}(t)$	1	0	0	0
$h_{01}(t)$	0	1	0	0
$h_{10}(t)$	0	0	1	0
$h_{11}(t)$	0	0	0	1

Hermite Cubic Basis

- Can satisfy with cubic polynomials as basis

$$h_{ij}(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0$$

- Obtain - solve 4 linear equations in 4 unknowns for each basis function

$$h_{ij}(t): i, j = 0,1, t \in [0,1]$$

curve	$C(0)$	$C(1)$	$C'(0)$	$C'(1)$
$h_{00}(t)$	1	0	0	0
$h_{01}(t)$	0	1	0	0
$h_{10}(t)$	0	0	1	0
$h_{11}(t)$	0	0	0	1

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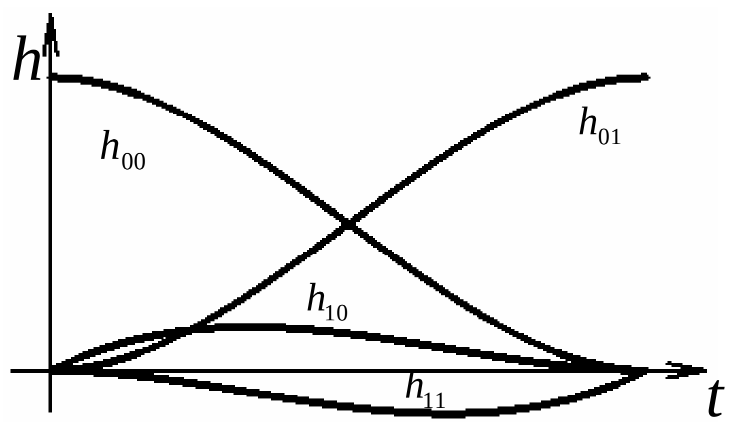
Hermite Cubic Basis

- Four polynomials that satisfy the conditions

$$h_{00}(t) = t^2(2t-3)+1 \qquad h_{01}(t) = -t^2(2t-3)$$

$$h_{10}(t) = t(t-1)^2 \qquad h_{11}(t) = t^2(t-1)$$



hermite

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Natural Cubic Splines

- Standard spline input – set of points $\{P_i\}_{i=0}^{n}$
  - No derivatives
- Interpolate by n cubic segments:
  - Derive $\{T_i\}_{i=0}^{n}$ from continuity constraints
  - Solve 4n equations

Interpolation ($2n$ equations):

$$C_i(0) = P_{i-1} \quad C_i(1) = P_i \quad i = 1,..,n$$

C^1 continuity constraints ($n-1$ equations):

$$C_i'(1) = C_{i+1}'(0) \qquad i = 1,..,n-1$$

C^2 continuity constraints ($n-1$ equations):

$$C_i''(1) = C_{i+1}''(0) \qquad i = 1,..,n-1$$

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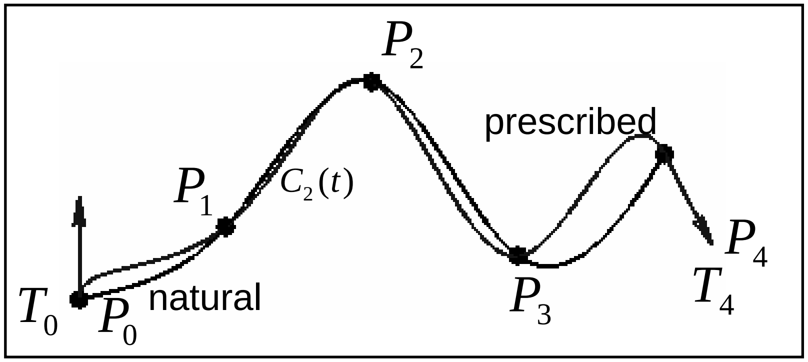
Natural Cubic Splines

- Need another 2 equations to reach 4n
- Options
  - Natural end conditions: $C_1''(0) = 0, C_n''(1) = 0$
  - Prescribed end conditions (derivative available): $C_1'(0) = T_0, C_n'(1) = T_n$



Freeform Representation - Surfaces

- Explicit form: z=z(x,y)
- Implicit form: f(x,y,z)=0
- Parametric form: [x(u,v),y(u,v),z(u,v)]
- Example – origin centered sphere of radius *R:*

Explicit is a special case of implicit and parametric form

Explicit:

$$z = +\sqrt{R^2 - x^2 - y^2} \cup z = -\sqrt{R^2 - x^2 - y^2}$$

Implicit:

$$x^2 + y^2 + z^2 - R^2 = 0$$

Parametric:

$$(x,y,z) = (R\cos\theta\cos\psi, R\sin\theta\cos\psi, R\sin\psi), \theta \in [0,2\pi], \psi \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

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From Curves to Surfaces – Tensor Splines

- Curve is expressed as inner product of P_i coefficients and basis functions

$$C(u) = \sum_{i=0}^{n} P_i B_i(u)$$

- To extend curves to surfaces - treat surface as a curve of curves
- Assume P_i is not constant, but a function of second parameter v: $P_i(v) = \sum_{j=0}^{m} Q_{ij} B_j(v)$

$$C(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} Q_{ij} B_j(v) B_i(u)$$

bezpatch

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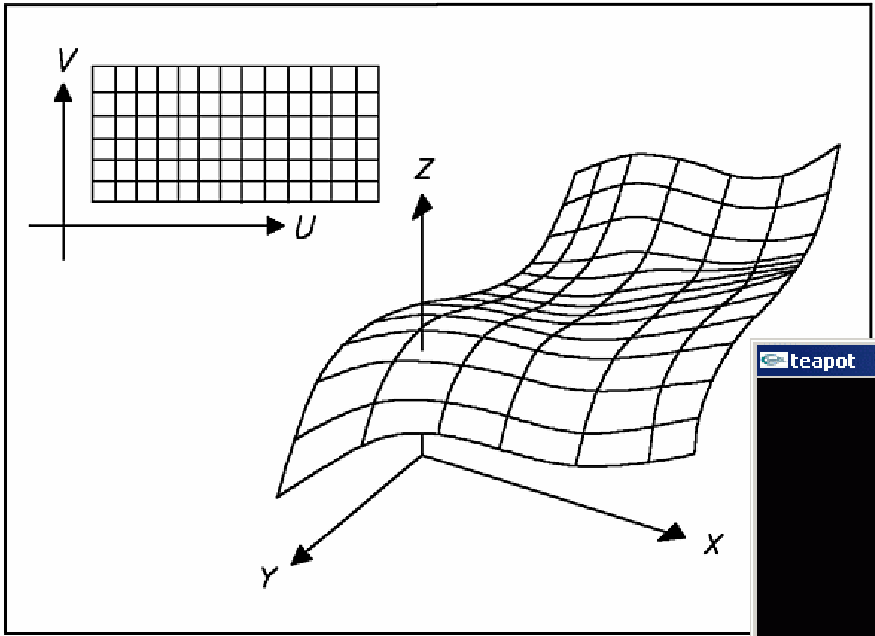
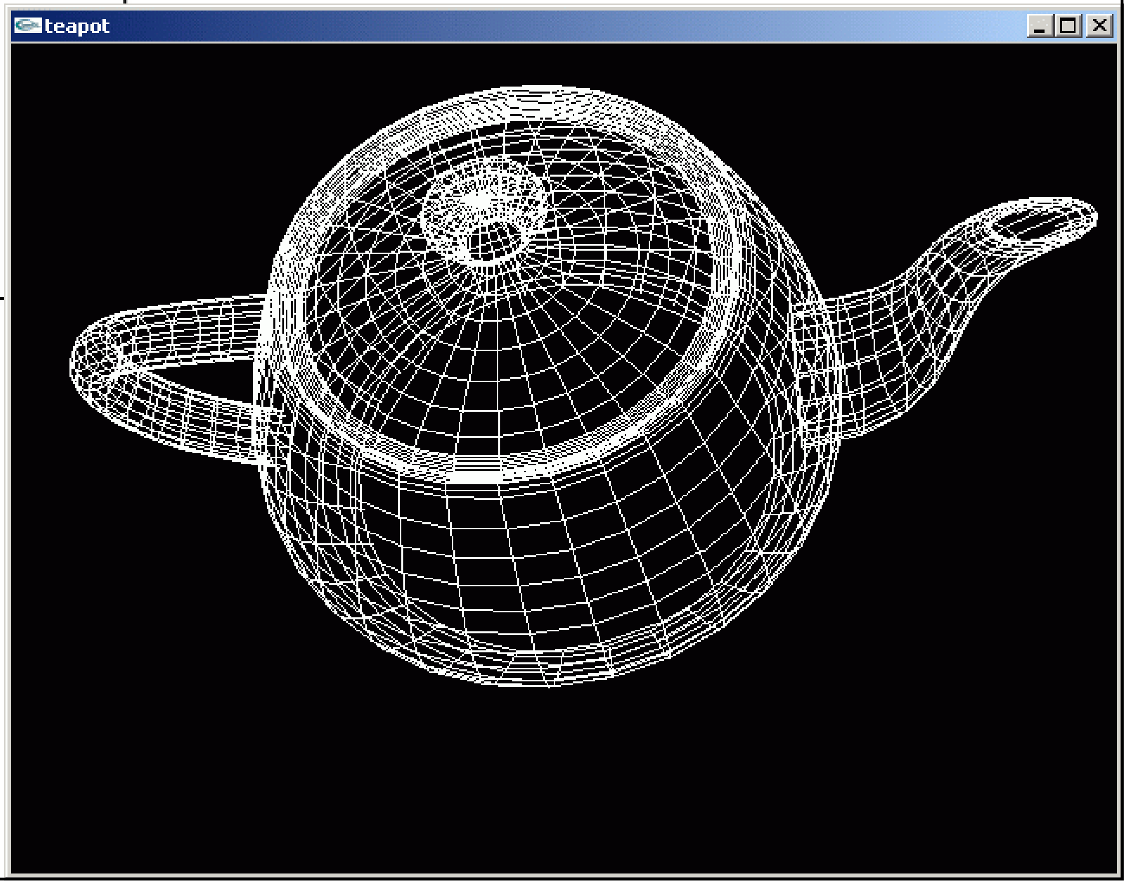
Tensor Spline Surfaces





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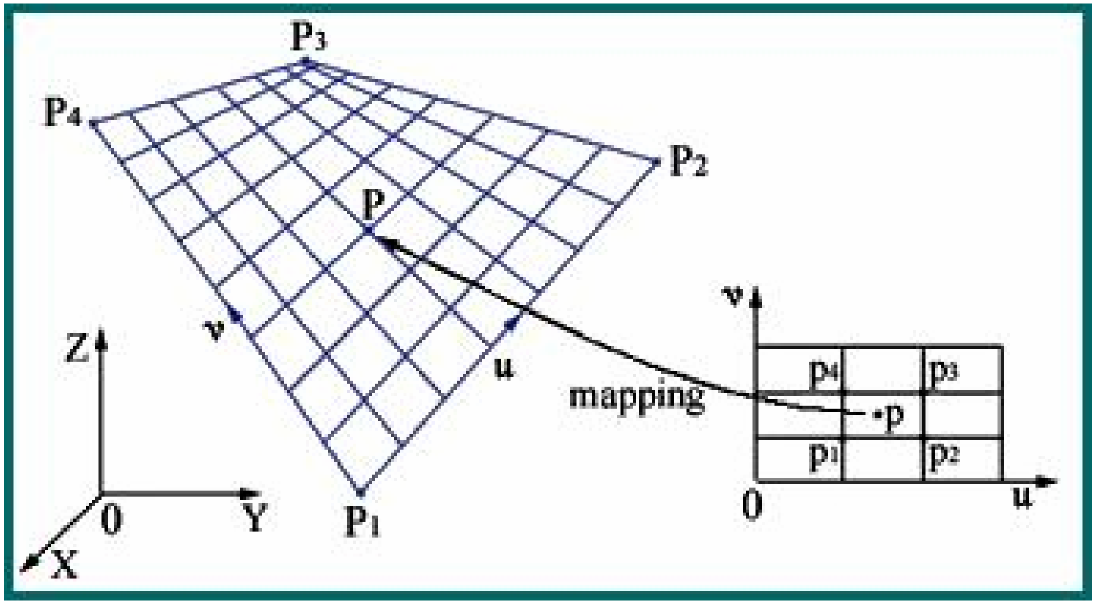
Bilinear Patches

- Bilinear interpolation of 4 3D points

$$P_{00}, P_{01}, P_{10}, P_{11}$$

- surface analog of line segment curve



Bilinear Patches

- Given $P_{00}, P_{01}, P_{10}, P_{11}$ associated parametric bilinear surface for $u, v \in [0,1]$ is:

$$P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11}$$

- Questions:
  - What does an isoparametric curve of a bilinear patch look like ?
  - When is a bilinear patch planar?

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