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Geometric Transformations

- *review of relevant math*
- *4x4 transformation matrices*

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Math Review

matrix vector multiplication

- points as column vectors

$$h=1 \quad \begin{bmatrix}x'\\y'\\z'\\h'\end{bmatrix}=\begin{bmatrix}m_{11} & m_{12} & m_{13} & m_{14}\\ m_{21} & m_{22} & m_{23} & m_{24}\\ m_{31} & m_{32} & m_{33} & m_{34}\\ m_{41} & m_{42} & m_{43} & m_{44}\end{bmatrix}\begin{bmatrix}x\\y\\z\\h\end{bmatrix} \quad h=1$$

$$P' = MP$$

- points as row vectors

$$\begin{bmatrix}x' & y' & z' & h'\end{bmatrix}=\begin{bmatrix}x & y & z & h\end{bmatrix}\begin{bmatrix}m_{11} & m_{12} & m_{13} & m_{14}\\ m_{21} & m_{22} & m_{23} & m_{24}\\ m_{31} & m_{32} & m_{33} & m_{34}\\ m_{41} & m_{42} & m_{43} & m_{44}\end{bmatrix}^T$$

$$P'^T = P^T M^T$$

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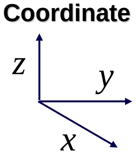
Math Review

Coordinate Systems

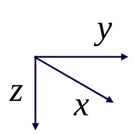
Right-handed Coordinate System



$z = x \times y$

using right-hand rule

Left-handed Coordinate System



$z = x \times y$

using left-hand rule

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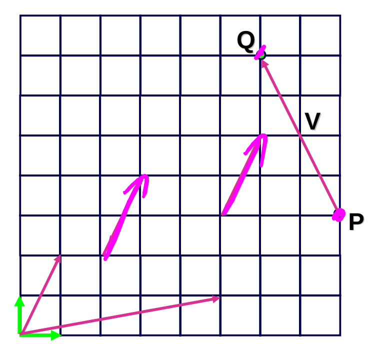
Math Reviw

Points and Vectors



vector space
vectors are invariant
under translation

affine space:
allows vector-to-point addition

translations matter for points

$$P + V = Q$$

$$Q - P = V$$

Math Reviw

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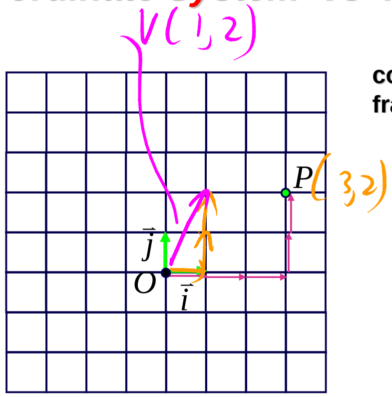
Coordinate System vs Frame



coordinate system: basis vectors
frame: basis vectors + Origin

allows for points

$V = x\vec{i} + y\vec{j}$

$P = O + x\vec{i} + y\vec{j} + z\vec{k}$

origin

Math Review

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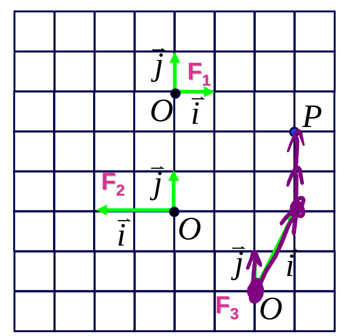
Working with Frames



$P = O + x\vec{i} + y\vec{j}$

F_1 **P(3,-1)** $O_1 + 3\vec{i}_1 - 1\vec{j}_1$

F_2 **P(-1.5,2)**

F_3 **P(1,2)** $1\vec{i} + 2\vec{j}$

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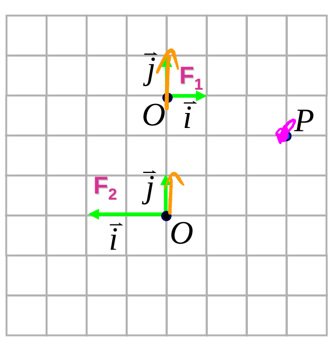
Transformations as a change of frame



$P = O + x\vec{i} + y\vec{j}$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_1 + x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}_1 + y_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}_1$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}_2 + x_1 \begin{bmatrix} -0.5 \\ 0 \end{bmatrix}_2 + y_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}_2$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_2 = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_1 \quad \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

check: P_1(3,-1) becomes P_2(-1.5,2)

$P_2 = MP_1$

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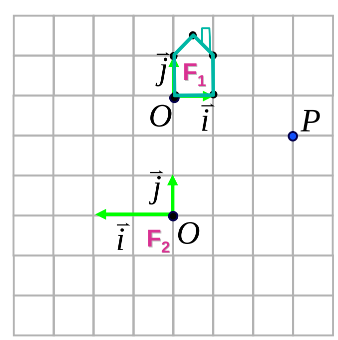
change of basis expressed using a matrix



$$\begin{bmatrix} x \\ y \end{bmatrix}_2 = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_1$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_2 = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_1$$

Usage of Transformations

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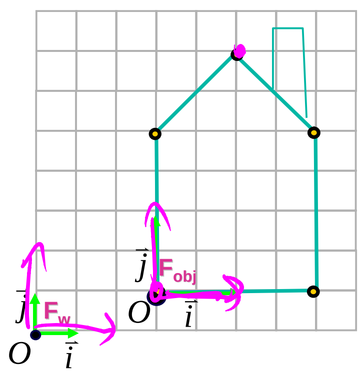

$$P = O + x\vec{i} + y\vec{j}$$

$$\begin{bmatrix} x_{obj} \\ y_{obj} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{obj} + x_{obj}\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{obj} + y_{obj}\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{obj}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_w = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_w + x_{obj}\begin{bmatrix} 2 \\ 0 \end{bmatrix}_w + y_{obj}\begin{bmatrix} 0 \\ 2 \end{bmatrix}_w$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_w = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_{obj}$$

Using Transformations

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$$\begin{bmatrix} x \\ y \end{bmatrix}_2 = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_1$$

2D

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_w = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_{obj}$$

3D

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_w = \begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{obj}$$

GLfloat T[16] = { 2,0,0,0, 0,2,0,0, 0,0,1,0 3,1,0,1};

Usage of Transformations

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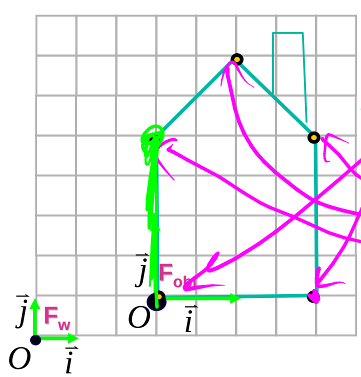


```
GLfloat T[16] = { ... };
glMatrixMode(GL_MODELVIEW);
glLoadMatrixf(T);

glBegin(GL_LINE_LOOP);
glVertex2f(0,0);
glVertex2f(2,0);
glVertex2f(2,2);
glVertex2f(1,3);
glVertex2f(0,2);
glEnd();
```


Usage of Transformations

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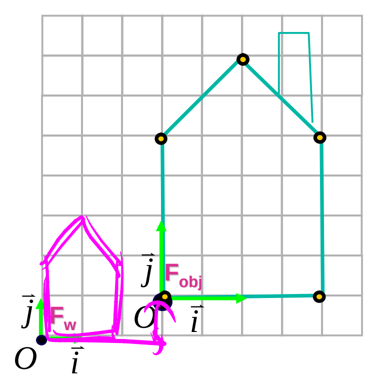
An easier way to do the same thing....

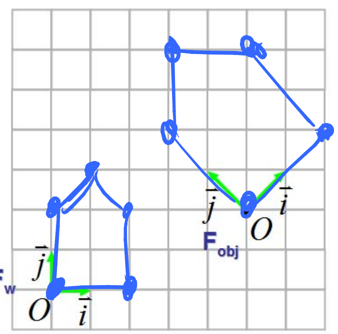



```
glMatrixMode(GL_MODELVIEW);
glLoadIdentity();

glTranslatef(3,1,0);
glScale(2,2,2);

glBegin(GL_LINE_LOOP);
glVertex2f(0,0);
glVertex2f(2,0);
glVertex2f(2,2);
glVertex2f(1,3);
glVertex2f(0,2);
glEnd();
```

T



P(0,0)
P(2,0)
P(2,2)
P(1,3)
P(0,2)

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_w = \begin{array}{ccc} i & j & \text{origin} \\ \end{array}\begin{bmatrix} 1 & -1 & 5 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_{obj}$$

Transformations

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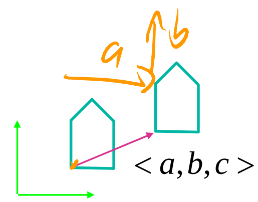
Translation



translate(a,b,c)

translations

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & & & a \\ & 1 & & b \\ & & 1 & c \\ & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$x' = 1 \cdot x + a$
$y' = 1 \cdot y + b$
$z' = 1 \cdot z + c$

glTranslatef(a,b,c);
glTranslated(a,b,c);

$P = \underbrace{M}P + \underbrace{T}$

Transformations

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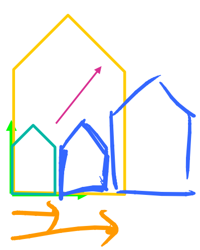
Scaling

scale(a,b,c)

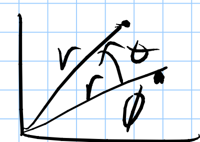


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

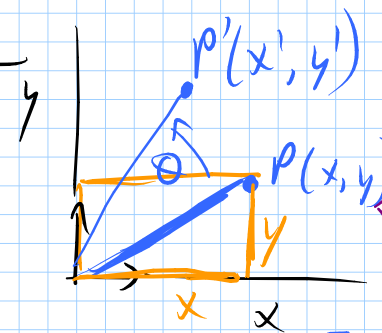
$x' = ax$
$y' = by$
$z' = cz$

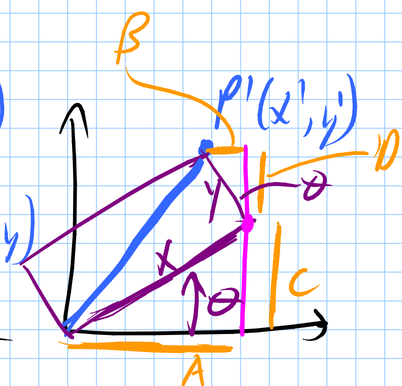
glScalef(a,b,c);
glScaled(a,b,c);

$\underbrace{a, a, a}$



$x' = r\cos(\theta + \phi)$
$y' = r\sin(\theta + \phi)$





$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$x' = A - B$
$= x\cos\theta - y\sin\theta$
$y' = C + D$
$= x\sin\theta + y\cos\theta$

Transformations

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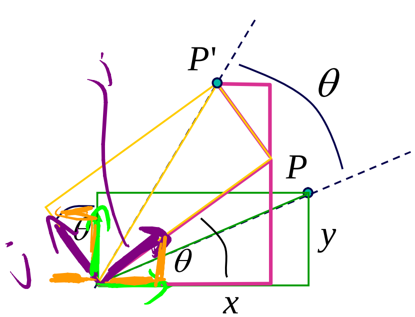
Rotation



$$Rotate(z,\theta)$$

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

$$z' = z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & \\ & & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

glRotatef(angle,x,y,z);
glRotated(angle,x,y,z); (30°, 0, 0, 1)

Transformations

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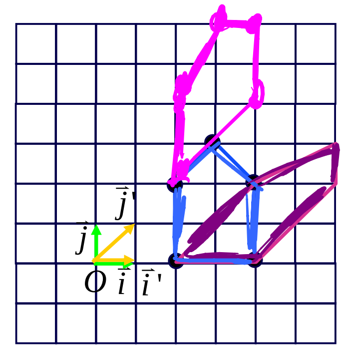
Shear



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & & \\ 0 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

shear

$x' = x + y$
$y' = y$
$z' = z$

$x' = x$
$y' = y + x$
$z' = z$

Transformations

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Affine transformations

- linear transformation + translations
- can be expressed as a 3x3 matrix + 3 vector

$$P' = M \cdot P + T$$

4x4 matrices

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & T_x \\ m_{21} & m_{22} & m_{23} & T_y \\ m_{31} & m_{32} & m_{33} & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

h=1

rotations
scales
shears

translations

Projective Rendering Pipeline

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glVertex3f(x,y,z)

4x4 4x4

OCS → **modeling transformation** → WCS → **viewing transformation** → VCS → **projection transformation** → CCS → **perspective division** → NDCS → **viewport transformation** → DCS

glTranslatef(x,y,z)
glRotatef(th,x,y,z)
....

gluLookAt(...)

glFrustum(...)

glutInitWindowSize(x,y)

OCS - object coordinate system
WCS - world coordinate system
VCS - viewing coordinate system
CCS - clipping coordinate system
NDCS - normalized device coordinate system
DCS - device coordinate system