University of British Columbia
CPSC 314 Computer Graphics
Jan-Apr 2005

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Modelling: Curves

Week 11, Wed Mar 23

http://www.ugrad.cs.ubc.ca/~cs314/Vjan2005

News

- reminder: my office hours today 3:45
- proposals due today 6pm

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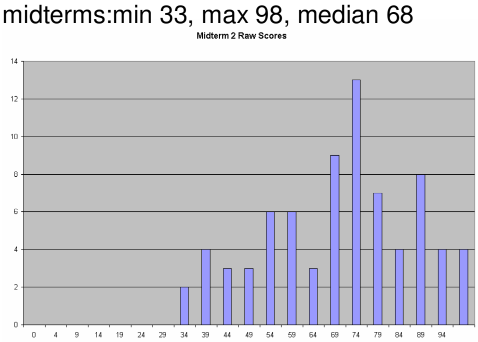
News

- midterms:min 33, max 98, median 68



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Reading

- FCG Chap 13

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Parametric Curves

- parametric form for a line:

$$x = x_0 t + (1-t)x_1$$
$$y = y_0 t + (1-t)y_1$$
$$z = z_0 t + (1-t)z_1$$

- x, y and z are each given by an equation that involves:
  - parameter *t*
  - some user specified control points, x_0 and x_1
- this is an example of a parametric curve

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Splines

- a *spline* is a parametric curve defined by *control points*
  - term "spline" dates from engineering drawing, where a spline was a piece of flexible wood used to draw smooth curves
  - control points are *adjusted by the user* to control shape of curve

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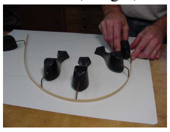
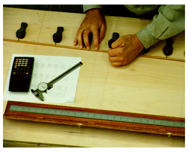
Splines - History

- draftsman used ‘ducks’ and strips of wood (splines) to draw curves
- wood splines have second-order continuity, pass through the control points

a duck (weight)



ducks trace out curve



Hermite Spline

- *hermite spline* is curve for which user provides:
  - endpoints of curve
  - parametric derivatives of curve at endpoints
    - parametric derivatives are *dx/dt, dy/dt, dz/dt*
  - more derivatives would be required for higher order curves

Hermite Cubic Splines

- example of knot and continuity constraints

Hermite Specification

Hermite Spline (2)

- say user provides x_0, x_1, x_0', x_1'
- cubic spline has degree 3, is of the form:

$$x = at^3 + bt^2 + ct + d$$

  - for some constants a, b, c and d derived from the control points, but how?
- we have constraints:
  - curve must pass through x_0 when $t=0$
  - derivative must be x'_0 when $t=0$
  - curve must pass through x_1 when $t=1$
  - derivative must be x'_1 when $t=1$

Hermite Spline (3)

- solving for the unknowns gives

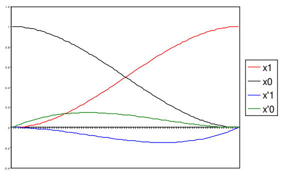
$$\begin{aligned} a &= -2x_1 + 2x_0 + x_1' + x_0' \\ b &= 3x_1 - 3x_0 - x_1' - 2x_0' \\ c &= x_0' \\ d &= x_0 \end{aligned}$$

- rearranging gives

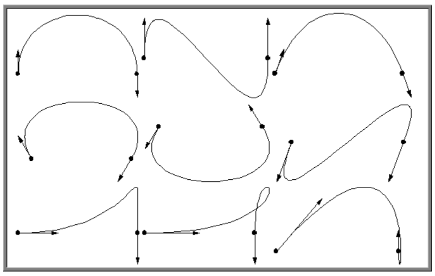
$$\begin{aligned} x = {} & x_1(-2t^3 + 3t^2) \\ & + x_0(2t^3 - 3t^2 + 1) \\ & + x_1'(t^3 - t^2) \\ & + x_0'(t^3 - 2t^2 + t) \end{aligned} \quad \text{or} \quad x = \begin{bmatrix} x_1 & x_0 & x_1' & x_0' \end{bmatrix} \begin{bmatrix} -2 & 3 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Basis Functions

- a point on a Hermite curve is obtained by multiplying each control point by some function and summing
- functions are called *basis functions*



Sample Hermite Curves



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Splines in 2D and 3D

- so far, defined only 1D splines: $x=f(t:x_0,x_1,x'_0,x'_1)$
- for higher dimensions, define control points in higher dimensions (that is, as vectors)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 & x_0 & x'_1 & x'_0 \\ y_1 & y_0 & y'_1 & y'_0 \\ z_1 & z_0 & z'_1 & z'_0 \end{bmatrix} \begin{bmatrix} -2 & 3 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

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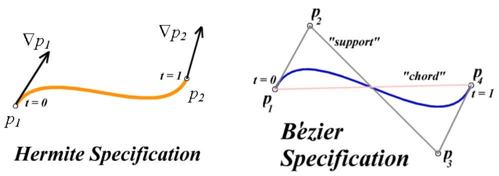
Bézier Curves

- similar to Hermite, but more intuitive definition of endpoint derivatives
- four control points, two of which are knots



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Bézier Curves

- derivative values of Bezier curve at knots dependent on adjacent points

$$\nabla p_1 = 3(p_2 - p_1)$$
$$\nabla p_4 = 3(p_4 - p_3)$$

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Bézier vs. Hermite

- can write Bezier in terms of Hermite
  - note: just matrix form of previous

$$\underbrace{\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \frac{dx_1}{dt} & \frac{dy_1}{dt} \\ \frac{dx_2}{dt} & \frac{dy_2}{dt} \end{bmatrix}}_{\mathbf{G}_{Hermite}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}}_{\mathbf{G}_{Bezier}}$$

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Bézier vs. Hermite

- Now substitute this in for previous Hermite

$$\begin{bmatrix} a_x & a_y \\ b_x & b_y \\ c_x & c_y \\ d_x & d_y \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{M}_{Hermite}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}}_{\mathbf{G}_{Bezier}}$$

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Bézier Basis, Geometry Matrices

$$\begin{bmatrix} a_x & a_y \\ b_x & b_y \\ c_x & c_y \\ d_x & d_y \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{M}_{Bezier}} \underbrace{\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}}_{\mathbf{G}_{Bezier}}$$

- but why is M_{Bezier} a good basis matrix?

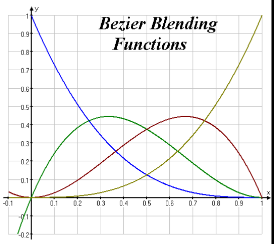
Bézier Blending Functions

- look at blending functions
- family of polynomials called order-3 Bernstein polynomials
  - C(3, k) t^k $(1-t)^{3-k}$; 0<= k <= 3
  - all positive in interval [0,1]
  - sum is equal to 1

$$p(t) = \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

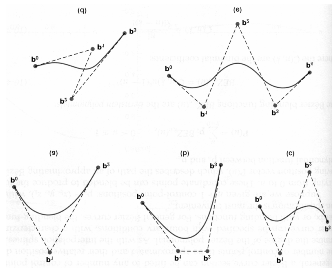
Bézier Blending Functions

- every point on curve is linear combination of control points
- weights of combination are all positive
- sum of weights is 1
- therefore, curve is a convex combination of the control points



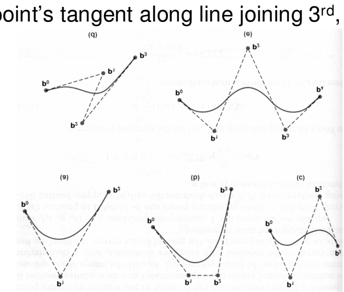
Bézier Curves

- curve will always remain within convex hull (bounding region) defined by control points



Bézier Curves

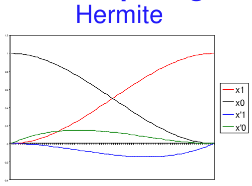
- interpolate between first, last control points
- 1st point's tangent along line joining 1st, 2nd pts
- 4th point's tangent along line joining 3rd, 4th pts

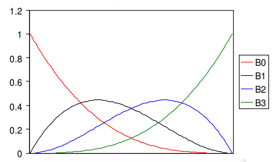


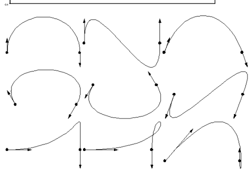
Comparing Hermite and Bézier

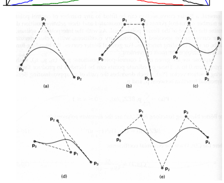
Hermite

Bézier



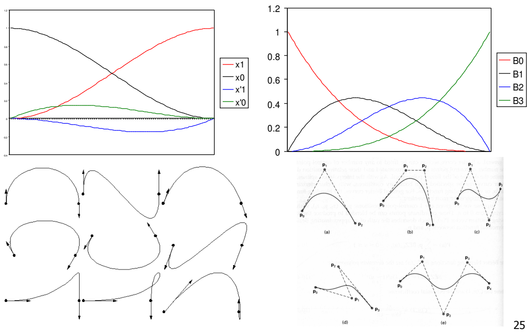






Comparing Hermite and Bezier

demo: www.siggraph.org/education/materials/HyperGraph/modeling/splines/demoprog/curve.html



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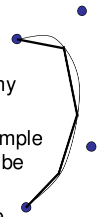
Rendering Bezier Curves: Simple

- evaluate curve at fixed set of parameter values, join points with straight lines
- advantage: very simple
- disadvantages:
  - expensive to evaluate the curve at many points
  - no easy way of knowing how fine to sample points, and maybe sampling rate must be different along curve
  - no easy way to adapt: hard to measure deviation of line segment from exact curve



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Rendering Beziers: Subdivision

- a cubic Bezier curve can be broken into two shorter cubic Bezier curves that exactly cover original curve
- suggests a rendering algorithm:
  - keep breaking curve into sub-curves
  - stop when control points of each sub-curve are nearly collinear
  - draw the control polygon: polygon formed by control points

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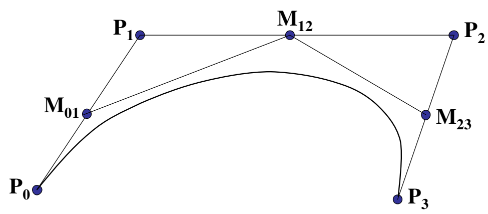
Sub-Dividing Bezier Curves

- step 1: find the midpoints of the lines joining the original control vertices. call them M_{01}, M_{12}, M_{23}



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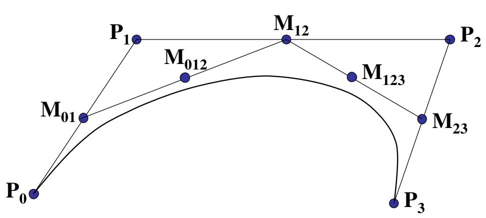
Sub-Dividing Bezier Curves

- step 2: find the midpoints of the lines joining M_{01}, M_{12} and M_{12}, M_{23}. call them M_{012}, M_{123}



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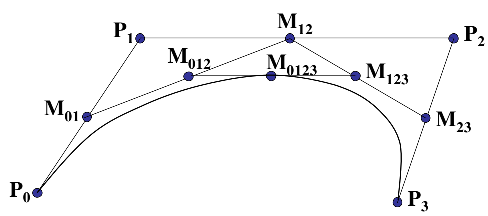
Sub-Dividing Bezier Curves

- step 3: find the midpoint of the line joining M_{012}, M_{123}. call it M_{0123}



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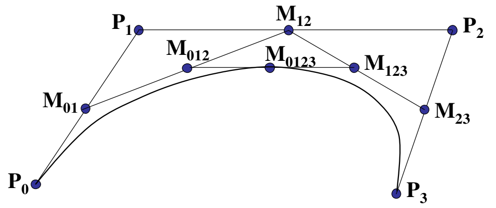
Sub-Dividing Bezier Curves

- curve P_0, M_{01}, M_{012}, M_{0123} exactly follows original from t=0 to t=0.5
- curve M_{0123}, M_{123}, M_{23}, P_3 exactly follows original from t=0.5 to t=1



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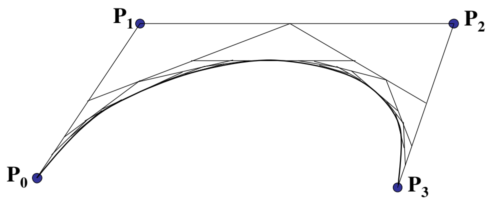
Sub-Dividing Bezier Curves

- continue process to create smooth curve



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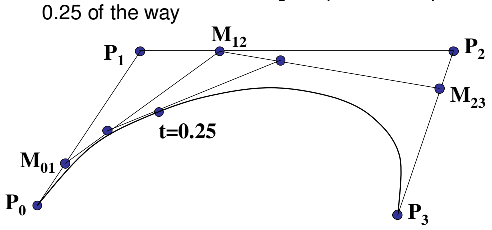
de Casteljau's Algorithm

- can find the point on a Bezier curve for any parameter value t with similar algorithm
  - for $t=0.25$, instead of taking midpoints take points 0.25 of the way



demo: www.saltire.com/applets/advanced_geometry/spline/spline.htm

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Longer Curves

- a single cubic Bezier or Hermite curve can only capture a small class of curves
  - at most 2 inflection points
- one solution is to raise the degree
  - allows more control, at the expense of more control points and higher degree polynomials
  - control is not local, one control point influences entire curve
- better solution is to join pieces of cubic curve together into piecewise cubic curves
  - total curve can be broken into pieces, each of which is cubic
  - local control: each control point only influences a limited part of the curve
  - interaction and design is much easier

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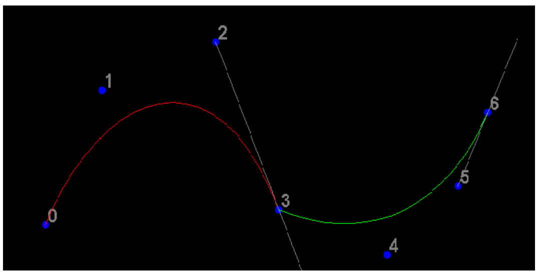
Piecewise Bezier: Continuity Problems



demo: www.cs.princeton.edu/~min/cs426/jar/bezier.html

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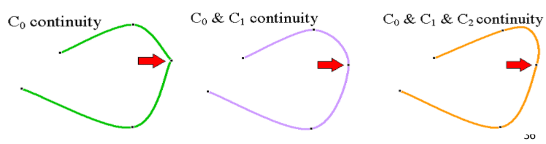
Continuity

- when two curves joined, typically want some degree of continuity across knot boundary
  - C0, "C-zero", point-wise continuous, curves share same point where they join
  - C1, "C-one", continuous derivatives
  - C2, "C-two", continuous second derivatives



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Geometric Continuity

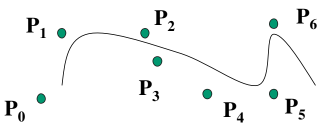
- derivative continuity is important for animation
  - if object moves along curve with constant parametric speed, should be no sudden jump at knots
- for other applications, *tangent continuity* suffices
  - requires that the tangents point in the same direction
  - referred to as G^1 *geometric continuity*
  - curves could be made C^1 with a re-parameterization
  - geometric version of C^2 is G^2, based on curves having the same radius of curvature across the knot

Achieving Continuity

- Hermite curves
  - user specifies derivatives, so C^1 by sharing points and derivatives across knot
- Bezier curves
  - they interpolate endpoints, so C^0 by sharing control pts
  - introduce additional constraints to get C^1
    - parametric derivative is a constant multiple of vector joining first/last 2 control points
    - so C^1 achieved by setting $P_{0,3}=P_{1,0}=J$, and making $P_{0,2}$ and J and $P_{1,1}$ collinear, with $J-P_{0,2}=P_{1,1}-J$
    - C^2 comes from further constraints on $P_{0,1}$ and $P_{1,2}$
  - leads to...

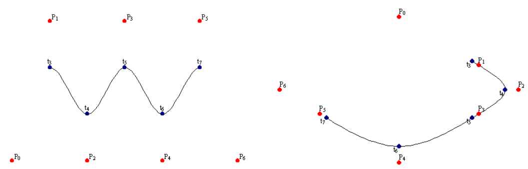
B-Spline Curve

- start with a sequence of control points
- select four from middle of sequence
  (p_{i-2}, p_{i-1}, p_i, p_{i+1})
  - Bezier and Hermite goes between p_{i-2} and p_{i+1}
  - B-Spline doesn't interpolate (touch) any of them but approximates the going through p_{i-1} and p_i



B-Spline

- by far the most popular spline used
- C_0, C_1, and C_2 continuous



demo: www.siggraph.org/education/materials/HyperGraph/modeling/splines/demoprog/curve.html

B-Spline

- locality of points

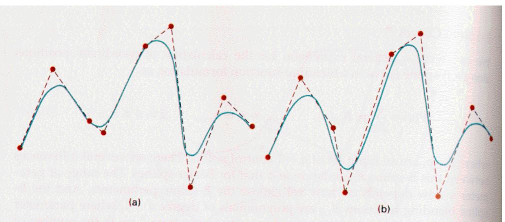


Figure 10-41
Local modification of a B-spline curve. Changing one of the control points in (a) produces curve (b), which is modified only in the neighborhood of the altered control point.