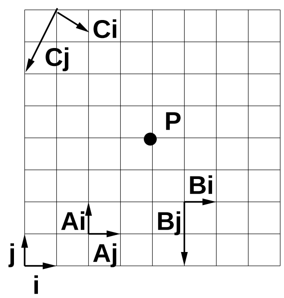
CPSC 314, Solution to Written Homework 1

Out: Fri 28 Jan 2005
Due: Fri 4 Feb 2005 4pm
Value: 5% of final grade
Total Points: 100

Transformations (50 pts)

1. (9 pts) The point coordinate P can be expressed as P = 4*i + 4*j, where i and j are basis vectors of unit length along the x and y axes, respectively. Describe the point P in terms of the 3 other coordinate systems given below.



Answer:

$$P_A = 3 * A_i + 2 * A_j \tag{1}$$

$$P_B = -1 * B_i - 1 * B_j \tag{2}$$

$$P_C = 4 * C_i + 1 * C_j \tag{3}$$

The above coordinates were obtained just by eye-balling the figure.

For an alternative way of expressing point P in term of coordinate Frame C, we can use the equations that convert from frame C to Frame A:

$$P_A = O_{C2A} + P_{Cx} * i_{C2A} + P_{Cy} * j_{C2A} \tag{4}$$

$$P_A = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, O_{C2A} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}, i_{C2A} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}, j_{C2A} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \tag{5}$$

Substitute back into system of equations:

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix} + P_{Cx} * \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + P_{Cy} * \begin{bmatrix} -2 \\ -1 \end{bmatrix} \tag{6}$$

Solve the system of 2 equations in 2 unknowns to get:

$$P_{Cx} = 4, P_{Cy} = 1 \tag{7}$$

Therefor,

$$P_C = 4 * C_i + 1 * C_j \tag{8}$$

2. (10 pts) Derive a transformation that takes a point from frame C to frame B. That is, determine M_{C2B}, where $P_B = M_{C2B}P_C$. Verify your solution using your answer to the question above.

Answer:

$$P_B = O_{C2B} + P_{Cx} * i_{C2B} + P_{Cy} * j_{C2B} \tag{9}$$

$$O_{C2B} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}, i_{C2B} = \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}, j_{C2B} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{10}$$

$$P_B = \begin{bmatrix} -4 \\ -3 \end{bmatrix} + P_{Cx} * \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} + P_{Cy} * \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{11}$$

Populate Transformation matrix, M_{C2B} with components of the above equation set:

$$M_{C2B} = \begin{bmatrix} 1 & -1 & -4 \\ \frac{1}{4} & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

Verify solution using answer to question 1:

$$P_B = M_{C2B}P_C = \begin{bmatrix} 1 & -1 & -4 \\ \frac{1}{4} & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

3. (3 pts) Write down the 4x4 matrix for translating an object by 2 in Z.

Answer:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. (3 pts) Write down the 4x4 matrix for nonuniformly scaling an object by 3 in X and 2 in Y.

Answer:

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5. (6 pts) Describe in words what this matrix does (be specific about the order of operations)

$$\begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Answer:

If you are thinking in terms of moving an object with respect to a global coordinate system, then:

This matrix first rotates an object by 90 degrees about the z axis and then translates the object -1 in the x-direction and -1 in the y-direction.

However, if you are thinking in a local system, then you are describing how the matrix transforms the entire coordinate frame. Here, the correct answer is:

First, the coordinate frame is translated by -1 in both the x and y directions, and then the coordinate frame is rotated by 90 degrees about z.

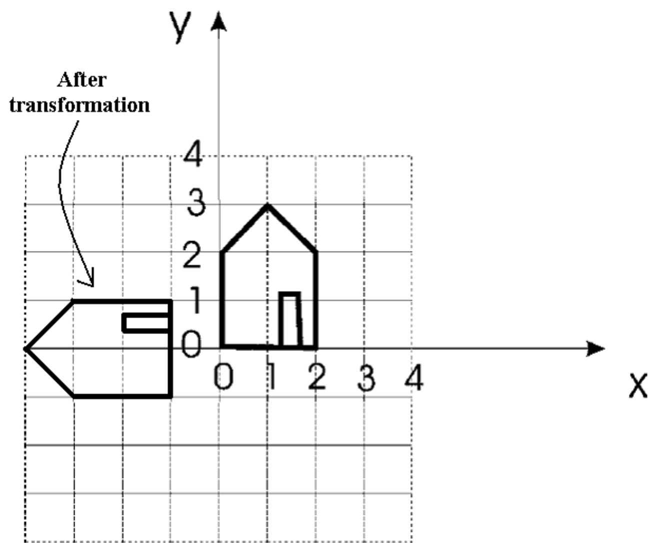
The following is a correct decomposition of the matrix. When thinking in terms of global coordinates, you keep applying the matrix that is closest to the point. i.e. you "read" the matrices from right to left. Locally, you "read" the matrices from left to right.

Either way, before discussion order of operations, you should state wether you are talking about global or local transformations.

$$\begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6. (5 pts) Draw a picture of the object below transformed by the above matrix

Answer:



7. (6 pts) Give the series of matrices needed to rotate a scene by 45° around the x axis with a fixed point of (1,2,3,1). Use column vectors for points, so that $p' = M_1 M_2 ... M_n p$.

Answer:

To rotate about a fixed point p, first translate p to the origin, rotate about the origin, then translate p back.

Translate P to the origin:

$$M_3 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$cos(45) = sin(45) = \frac{\sqrt{2}}{2}$

Rotate about the origin:

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(45) & -sin(45) & 0 \\ 0 & sin(45) & cos(45) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translate P back:

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

8. (5 pts) Give the sequence of OpenGL commands necessary to implement the above transformation.

Answer:

glTranslatef(1,2,3);

glRotatef(45,1,0,0);

glTranslatef(-1,-2,-3);

9. (3 pts) Normalize the homogeneous point (3,2,5,6).

   Answer:

   To normalize a homogeneous point, simply divide all components by the last component:

   (3/6,2/6,5/6,6/6) = (1/2,1/3,5/6,1)

Viewing (50 pts)

10. (4 pts) Give the viewing transformation matrix for an eye position (7,2,0), a lookat point (-1,-1, 0) and an up vector (0,0,1).

    Answer:

    Define $g = LookAt - Eye = (-8, -3, 0)^T$
    Define $w = -g/||g|| = (0.9363, 0.3511, 0)^T$
    Define $u = Up \times w/||Up \times w|| = (-0.3511, 0.9363, 0)^T$
    Define $v = w \times u = (0, 0, 1)^T$

$$M = \begin{bmatrix} u_1 & u_2 & u_3 & 0 \\ v_1 & v_2 & v_3 & 0 \\ w_1 & w_2 & w_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -e_x \\ 0 & 1 & 0 & -e_y \\ 0 & 0 & 1 & -e_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -.3511 & .9363 & 0 & .5852 \\ 0 & 0 & 1 & 0 \\ .9636 & .3511 & 0 & -7.2565 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

11. (4 pts) Give the perspective projection matrix with a near plane of 1, far plane of 20, right plane 10, left plane -8, top plane 11, and bottom plane -9.

    Answer:

$$M = \begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1/9 & 0 & 1/9 & 0 \\ 0 & 1/10 & 1/10 & 0 \\ 0 & 0 & -21/19 & -40/19 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

12. (6 pts) Give the NDC-to-viewport transformation matrix for a viewport 100 pixels wide and 50 pixels high, where that window's upper left corner is at location 200,300 from the origin at the upper left of the display.

    Answer:

    Note: Because the screen origin is at the top left we first need to flip the y coordinates.

$$M = \begin{bmatrix} 1 & 0 & 200 \\ 0 & 1 & 300 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{100-1}{2} \\ 0 & 1 & \frac{50-1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 100/2 & 0 & 0 \\ 0 & 50/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 50 & 0 & 249.5 \\ 0 & -25 & 324.5 \\ 0 & 0 & 1 \end{bmatrix}$$

13. (6 pts) A unit square has points A=(0,0,0,1), B=(0,1,0,1), C=(0,1,1,1), D=(0,0,1,1) in world coordinates. Give the coordinates of these four points in the camera coordinate system, after the viewing transformation above has been applied.

    Answer:

    $A' = (.5852, 0, -7.2566, 1)^T$
    $B' = (1.5215, 0, -6.9054, 1)^T$
    $C' = (1.5215, 1, -6.9054, 1)^T$
    $D' = (.5852, 1, -7.2566, 1)^T$

14. (6 pts) Then give the coordinates of these points in the normalized device coordinate system, after the perspective transformation above has been applied.

    $A'' = (-.7413, -.7257, 5.9151, 7.2566)^T \rightarrow (-.1022, -.1, .8151)^T$
    $B'' = (-.5982, -.6905, 5.5271, 6.9054)^T \rightarrow (-.0866, -.1, .8004)^T$
    $C'' = (-.5982, -.5905, 5.5271, 6.9054)^T \rightarrow (-.0866, -.0855, .8004)^T$
    $D'' = (-.7413, -.6257, 5.9151, 7.2566)^T \rightarrow (-.1022, -.0862, .8151)^T$

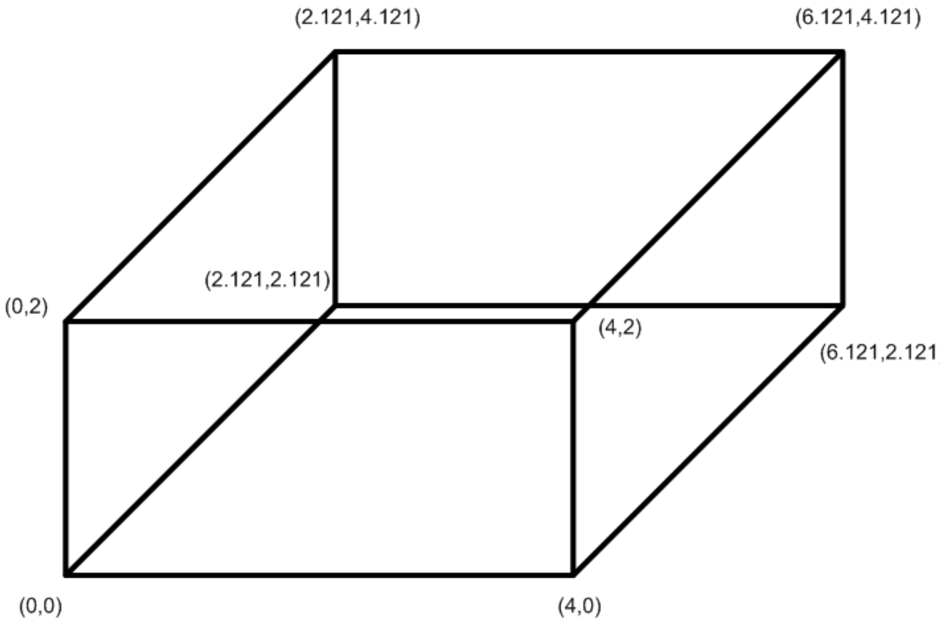
15. (6 pts) Finally, give the point coordinates in the display coordinate system described above, after the viewport transformation.

    $A''' = (244.3925, 327, 1)^T \rightarrow (244.3925, 327, 1)^T$
    $B''' = (245.1685, 327, 1)^T \rightarrow (245.1685, 327, 1)^T$

$C''' = (245.0338, 327, 1)^T \rightarrow (245.1685, 326.6380, 1)^T$
$D''' = (244.1502, 327, 1)^T \rightarrow (244.3925, 326.6555, 1)^T$

16. (8 pts) Draw a cavalier projection of a cube of size x=4, y=2, z=3. Use a 45° projection (that is, the z axis in the scene should make a 45° angle with the x axis in the projection). Label the points in your drawing with (x,y) locations.

Answer:



17. (10 pts) Give the 4x4 matrix that would produce the above cavalier projection. Hint: remember to ensure that points in the xy plane are not changed by the projection.

Answer:

$$M = \begin{bmatrix} 1 & 0 & \sqrt{2}/2 & 0 \\ 0 & 1 & \sqrt{2}/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$