Numerical Approximation and Discretization
CPSC 303, Term 2, Winter 2005–2006

Notes for Lecture 32 by Kang Yin

Introduction to Numerical Integration and Basic Quadrature

- **Examples of Clenshaw-Curtis Quadrature**

  Chebychev Polynomials:

$$\begin{aligned} x &\in [-1,1], \\ T_0(x) &= \cos(0) = 1, \\ T_1(x) &= \cos(\arccos x) = x, \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x). \end{aligned}$$

$$T_2(x) = 2x^2 - 1 \Longrightarrow x = \pm\sqrt{\frac{1}{2}}.$$

$$T_3(x) = 4x^3 - 3x \Longrightarrow x = 0, \pm\frac{\sqrt{3}}{2}.$$

  Two-point Clenshaw-Curtis Quadrature rule on $[-1,1]$:

$$x_0 = -\sqrt{\frac{1}{2}}, \quad x_1 = \sqrt{\frac{1}{2}},$$

$$a_0 = \int_{-1}^{1} \frac{x - \sqrt{\frac{1}{2}}}{-\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{2}}}\, dx = 1,$$

$$a_1 = \int_{-1}^{1} \frac{x + \sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{2}}}\, dx = 1,$$

$$I(f) = \int_{-1}^{1} f(x)\, dx \approx f(-\sqrt{\frac{1}{2}}) + f(\sqrt{\frac{1}{2}}).$$

  Three-point Clenshaw-Curtis Quadrature rule on $[-1,1]$:

$$x_0 = -\frac{\sqrt{3}}{2}, \quad x_1 = 0, \quad x_2 = \frac{\sqrt{3}}{2},$$

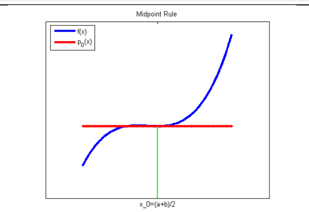
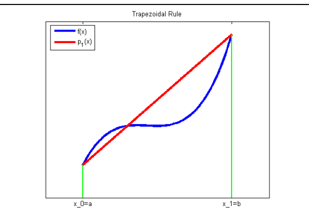
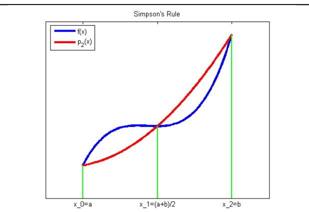
$$a_0 = \int_{-1}^{1} \frac{x(x - \frac{\sqrt{3}}{2})}{-\frac{\sqrt{3}}{2}(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2})}\, dx = \frac{4}{9},$$

$$a_1 = \int_{-1}^{1} \frac{(x + \frac{\sqrt{3}}{2})(x - \frac{\sqrt{3}}{2})}{-\frac{\sqrt{3}}{2}(-\frac{\sqrt{3}}{2})}\, dx = \frac{10}{9},$$

$$a_2 = \int_{-1}^{1} \frac{(x + \frac{\sqrt{3}}{2})x}{(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2})\frac{\sqrt{3}}{2}}\, dx = \frac{4}{9},$$

$$I(f) = \int_{-1}^{1} f(x)\, dx \approx \frac{4}{9} f(-\frac{\sqrt{3}}{2}) + \frac{10}{9} f(0) + \frac{4}{9} f(\frac{\sqrt{3}}{2}).$$

- **Table of Basic Quadrature Rules**

Quadrature	Illustration	$I(f)$	$E(f)$	precision
Midpoint		$(b-a)f(\frac{a+b}{2})$	$\frac{f''(\xi)}{24}(b-a)^3$	1
Trapezoidal		$\frac{b-a}{2}[f(a)+f(b)]$	$-\frac{f''(\xi)}{12}(b-a)^3$	1
Simpson		$\frac{b-a}{6}[f(a)+4f(\frac{a+b}{2})+f(b)]$	$-\frac{f''''(\xi)}{90}(\frac{b-a}{2})^5$	3