Unit #9: Graphs

CPSC 221: Basic Algorithms and Data Structures

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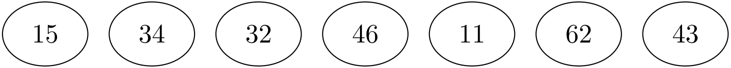
Unit Outline

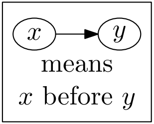
- Topological Sort: Sorting vertices
- Graph ADT and Graph Representations
- Graph Terminology
- More Graph Algorithms
  - Shortest Path (Dijkstra's Algorithm)
  - Minimum Spanning Tree (Kruskal's Algorithm)

Learning Goals

- Describe the properties and possible applications of various kinds of graphs (e.g., simple, complete), and the relationships among vertices, edges, and degrees.
- Prove basic theorems about simple graphs (e.g., handshaking theorem).
- Convert between adjacency matrices/lists and their corresponding graphs.
- Determine whether two graphs are isomorphic.
- Determine whether a given graph is a subgraph of another.
- Perform breadth-first and depth-first searches in graphs.
- Execute Dijkstra's shortest path algorithm and Kruskal's minimum spanning tree algorithm on a given graph.

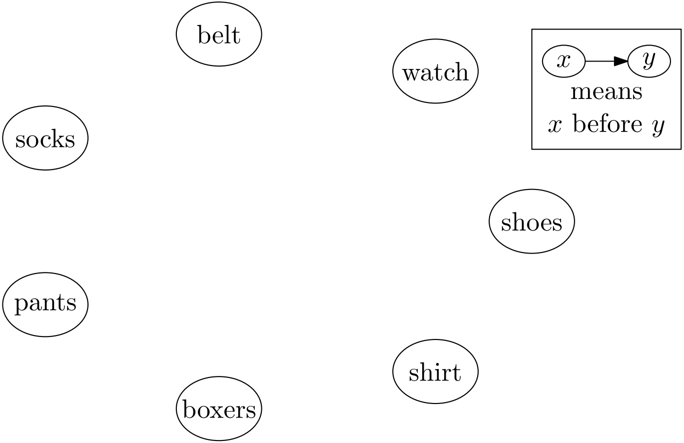
Sorting Total Orders





What property does the comparison-based sorting algorithm need to achieve?

Partial Order: Getting Dressed



Topological Sort

A topological sort is a total order of the vertices of a directed acyclic graph (DAG) $G = (V, E)$ such that if (u, v) is an edge of G then u appears before v in the order.

Topological Sort Algorithm I

Let $n = \#$ of vertices, $m = \#$ of edges, and $V =$ set of all vertices.

1. Find each vertex's *in-degree* ($\#$ of inbound edges).
2. While there are vertices remaining:
   - 2.1 Pick a vertex $v \in V$ with in-degree zero and output it.
   - 2.2 Reduce the in-degree of all vertices that v has an edge to.
   - 2.3 Remove v from the list of vertices.

Runtime?

Topological Sort Algorithm II

Let $n = \#$ of vertices, $m = \#$ of edges, and $V =$ set of all vertices.

1. Find each vertex's in-degree.
2. Initialize a queue to contain all in-degree zero vertices.
3. While there are vertices in the queue:
   3.1 Dequeue a vertex v (with in-degree zero) and output it.
   3.2 Reduce the in-degree of all vertices that v has an edge to.
   3.3 Enqueue any of these vertices that now have in-degree zero.

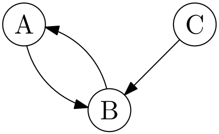
Runtime?

Graph ADT

A graph is a useful formalism for representing relationships among things.

A graph is represented as a pair of sets: $G = (V, E)$ where $|V| = n$ and $|E| = m$.

- V is a set of vertices: $\{v_1, v_2, \ldots, v_n\}$.
- E is a set of edges: $\{e_1, e_2, \ldots, e_m\}$ where each e_i is a pair of vertices: $e_i \in V \times V$.



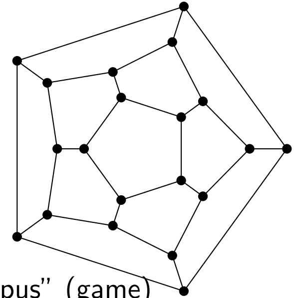
$V = \{A, B, C\}$

$E = \{(A, B), (B, A), (C, B)\}$

Operations may include:

- Create a graph (with a certain number of vertices).
- Insert or delete a given edge or vertex.
- Iterate over vertices adjacent to a given vertex.
- Ask if an edge exists that connects two given vertices.

Graph Applications



Storing things that are graphs by nature:

- Road networks
- Airline flights
- Relationships among people/things
- Room connections in “Hunt the Wumpus” (game)

Compilers:

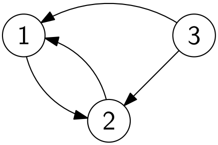
- Call graph - Which functions call other functions?
- Control flow graph - Which fragments of code can follow others?
- Dependency graphs - Which variables depend on others?

Others:

- Circuits, class hierarchies, meshes, networks of computers, ...

Graph Representation Using an Adjacency Matrix

A $|V| \times |V|$ array A where $A[u, v] = 1$ if and only if $(u, v) \in E$.



	1	2	3
1			
2			

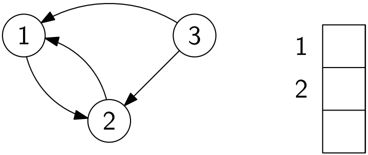
The runtime to:

- Iterate over n vertices is:
- Iterate over m edges (in a graph with n vertices) is:
- Iterate over all vertices adjacent to a vertex v is:
- Check whether an edge (u, v) exists is:

Memory requirements:

Graph Representation Using an Adjacency List

Adjacency List: An array L of $|V|$ lists, such that $L[u]$ contains v if and only if $(u, v) \in E$.



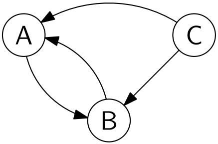
The runtime to:

- Iterate over n vertices is:
- Iterate over m edges is:
- Iterate over all vertices adjacent to a vertex v is:
- Check whether an edge (u, v) exists is:

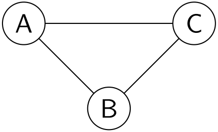
Memory requirements:

Directed vs. Undirected Graphs

In **directed** graphs, edges have a specific direction:



In **undirected** graphs, they don't (edges are two-way):

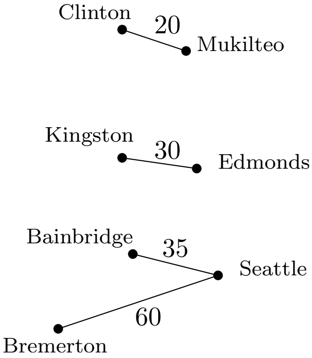


Vertices u and v are **adjacent** if $(u, v) \in E$.

What property do adjacency matrices of undirected graphs have?

Weighted Graphs

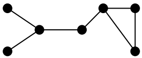
Each edge has an associated weight or cost. For example:



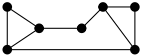
How can we store weights in an adjacency matrix?

In an adjacency list?

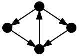
Graph Connectivity



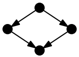
Connected: undirected and there is a path between any two vertices.



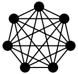
Biconnected: connected even after removing one vertex.



Strongly connected: directed and there is a path from any one vertex to any other.



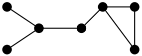
Weakly connected: directed and there is a path between any two vertices, ignoring direction.

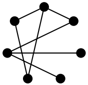


Complete graph: an edge between every pair of vertices

Isomorphism and Subgraphs

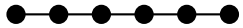
Isomorphic: Two graphs are isomorphic if they have the same structure (ignoring vertex names).

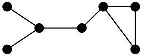




$G_1 = (V_1, E_1)$ is isomorphic to $G_2 = (V_2, E_2)$ if there is a one-to-one and onto function (i.e., bijection) $f : V_1 \to V_2$ such that $(u, v) \in E_1$ iff $(f(u), f(v)) \in E_2$.

Subgraph: One graph is a subgraph of another if it is some part of the other graph.





$G_1 = (V_1, E_1)$ is a subgraph of $G_2 = (V_2, E_2)$ if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$.
Note: We sometimes say that H is a subgraph of G if H is isomorphic to a subgraph (in the above sense) of G.

Degree

The degree of a vertex $v \in V$ is denoted $\deg(v)$ and represents the number of edges incident on v. (An edge from v to itself contributes 2 towards the degree.)

Handshaking Theorem:

If $G = (V, E)$ is an undirected graph, then

$$\sum_{v \in V} \deg(v) = 2|E|$$

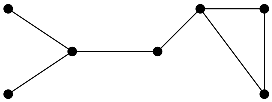
Corollary

An undirected graph has an even number of vertices of odd degree.

Degree/Handshake Example

The degree of a vertex $v \in V$ is the number of edges incident on v.

Let's label each vertex with its degree and calculate the sum:



Note that an edge contributes one to the degree of each endpoint.

Degree for Directed Graphs

The **in-degree** of a vertex $v \in V$ (denoted $\deg^-(v)$) is the number of edges coming in to v.

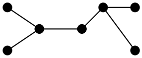
The **out-degree** of a vertex $v \in V$ (denoted $\deg^+(v)$) is the number of edges going out of v.

So, $\deg(v) = \deg^+(v) + \deg^-(v)$, and

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = \frac{1}{2} \sum_{v \in V} \deg(v).$$

Trees as Graphs

Tree: A tree is a connected, acyclic, undirected graph.



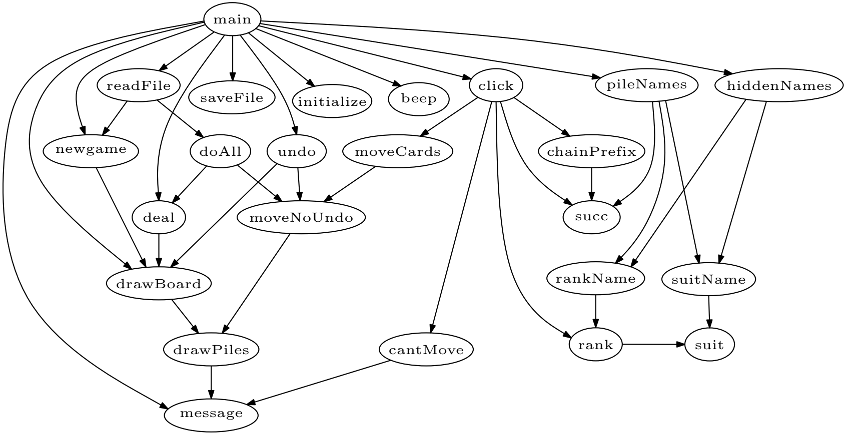
The number of edges $m = n - 1$ where n is the number of vertices.

Rooted tree: A rooted tree is a tree with a single distinguished vertex called the root.

We can imagine directing the edges of a rooted tree away from the root, to form a connected, acyclic, directed graph, in which there is a path from the root to every vertex.

Directed Acyclic Graphs (DAGs)

DAGs are directed graphs with no cycles.



We can topo-sort DAGs.

Single Source, Shortest Path Graphs

Given a graph $G = (V, E)$ and a vertex $s \in V$, find the shortest path from s to every vertex in V. The length of the path is the number of edges in the path.

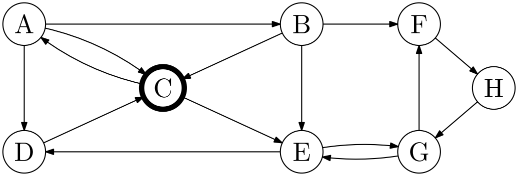
Many variations:

- Weighted vs. unweighted edges
- No cycles vs. cycles allowed
- Positive weights vs. negative weights allowed

Unweighted Single-Source Shortest Path Problem

```
BreadthFirstSearch(G, s)
  Q.enqueue([s,0])
  while Q is not empty:
      [v,d] = Q.dequeue()
      if v is unmarked:
        mark v with distance d
        for each edge (v,w):
            Q.enqueue([w,d+1])
```

(Replace the queue with a stack to get a depth-first search.)



General Breadth-First Search (BFS) Algorithm

```
BFS(v) using a starting vertex s in graph G:

  Add s to queue.
  While queue not empty:
       Dequeue vertex v.
       Process (e.g., print) v.  (If in search mode,
          return the information when the target is
          found, and terminate the algorithm.)
       Enqueue all unvisited neighbours of v.
```

We need to mark each vertex as being visited (so far), or not.

For a directed graph, u is a neighbour of v if (v, u) is an edge.

Application: Model a maze as a graph, and use BFS to find the shortest path/solution. (Koffman, p. 727+).

General Depth-First Search (DFS) Algorithm

```
DFS(v) using a starting vertex v in graph G:
    Call DFS(v)

DFS(v):
  Process (e.g., print) v.  (If in search mode, return
         the information when the target is found, and
         terminate the algorithm.)
  For each unvisited neighbour c of v:
       DFS(c)
```

We need to mark each vertex as being visited (so far), or not.

Application: Model a course prerequisite chart as a graph, and perform a topological sort (see Koffman, p. 731+).

Application: Solve a maze.

Weighted Single-Source Shortest Path

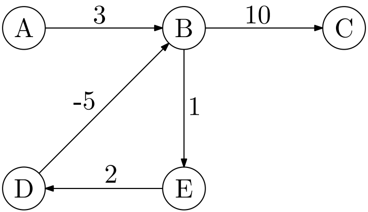
Assumes edge weights are non-negative.

Dijkstra's algorithm is a **greedy algorithm** (makes the current best choice without considering future consequences).

Intuition: Find the shortest paths in order of length.

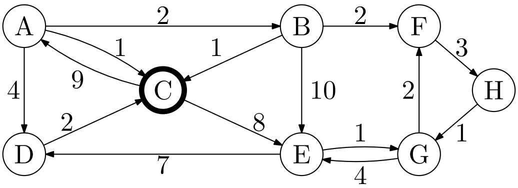
- Start at the source vertex (shortest path length $= 0$).
- The next shortest path extends some already discovered shortest path by one edge.
- Find it (by considering all one-edge extensions) and repeat.

The Trouble with Negative Weight Cycles



What's the shortest path from A to B (or C or D or E)?

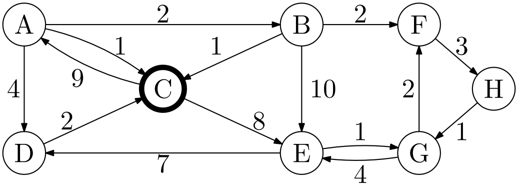
Intuition in Action



Dijkstra's Algorithm Pseudocode

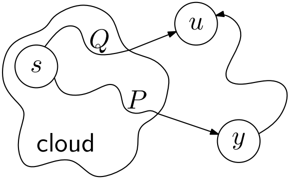
- Initialize the distance (dist) to each vertex to ∞.
- Initialize the dist to the source to 0.
- While there are unmarked vertices left in the graph:
  - Select the unmarked vertex v with the lowest dist.
  - Mark v with distance dist.
  - For each edge (v, w):
    - $\text{dist}(w) = \min\{\text{dist}(w), \text{dist}(v) + \text{weight of } (v, w)\}$

Dijkstra's Algorithm in Action



vertex	A	B	C	D	E	F	G	H
dist								
distance								

The Cloud Proof (by Contradiction)



- Assume Dijkstra's algorithm finds the correct shortest path to the first k vertices it visits (the **cloud**).
- But it fails on the $(k+1)$st vertex u.
- So there is some shorter path, P, from s to u.
- Path P must contain a first vertex y not in the cloud.
- But since the path, Q, to u is the shortest path out of the cloud, the path on P up to y must be at least as long as Q.
- Thus, the whole path P is at least as long as Q. Contradiction

(What did I use in that last step?)

Data Structures for Dijkstra's Algorithm

$n = |V|$ times: Select the unknown vertex with the lowest dist.
findMin/deleteMin

$m = |E|$ times: $\text{dist}(w) = \min\{\text{dist}(w), \text{dist}(v) + \text{weight of } (v, w)\}$
decreaseKey (i.e., change a key and fix the heap)
find by name (dictionary lookup)

Runtime: (Adjacency matrix or adjacency list?)

Fibonacci Heaps

- Very cool variation on Priority Queues
- Amortized $O(1)$ time for decreaseKey
- $O(\log n)$ time for deleteMin

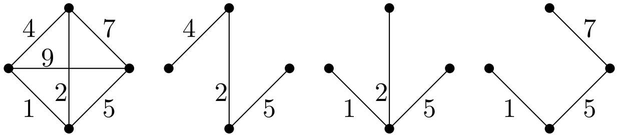
Dijkstra's Algorithm uses $n = |V|$ deleteMins and $m = |E|$ decreaseKeys.

Runtime with Fibonacci heaps:

Spanning Tree

Spanning tree: a subset of the edges from a connected graph that:

- touches all vertices in the graph (spans the graph), and
- forms a tree (is connected and contains no cycles)



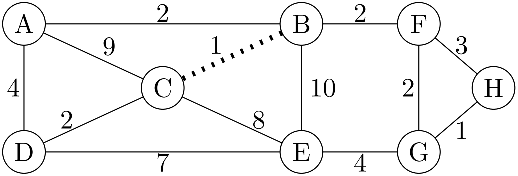
Minimum spanning tree: the spanning tree with the least total edge dist

Kruskal's Algorithm for Minimum Spanning Trees

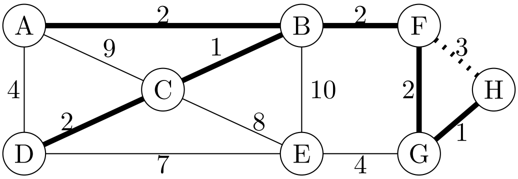
Yet another greedy algorithm:

- Start with an empty tree T.
- Repeat: Add the minimum weight edge to T **unless** it forms a cycle.

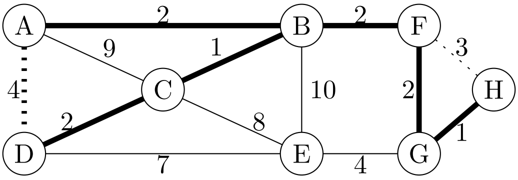
Kruskal's Algorithm in Action (1/5)



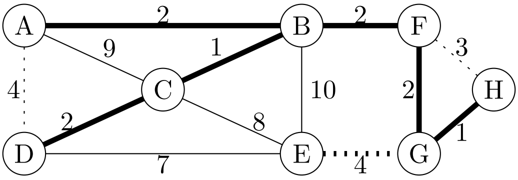
Kruskal's Algorithm in Action (2/5)



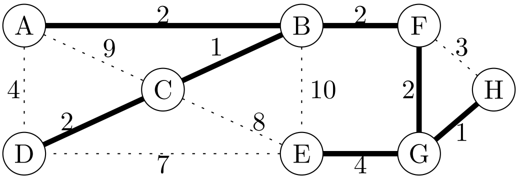
Kruskal's Algorithm in Action (3/5)



Kruskal's Algorithm in Action (4/5)



Kruskal's Algorithm Completed (5/5)



Proof of Correctness

Part I: Kruskal's Algorithm finds a spanning tree. Why?

Part II: Kruskal's Algorithm finds a minimum one.
Proof by contradiction.
Assume another spanning tree, T, has lower cost than Kruskal's tree K. (Pick T to be as similar to Kruskal's as possible.)
Pick an edge $e = (u, v)$ in T that's not in K.
Kruskal's Algorithm already rejected e because u and v were already connected by lesser (or equal) weight edges.
Take e out of T and add one of these lesser-weight edges to make a new spanning tree. Why does this work?
The new spanning tree still has lower cost than K and it's more like K. Contradiction.

Data Structures for Kruskal's Algorithm

$|E|$ times: Pick the lowest cost edge.
findMin/deleteMin

$|E|$ times: If u and v are not already connected, connect them.
find representative
union

With "disjoint-set" data structure, $O(|E| \log |E|)$ time.