Unit #2: Priority Queues

CPSC 221: Basic Algorithms and Data Structures

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Unit Outline

- Rooted Trees (Briefly)
- Priority Queue ADT
- Heaps
  - Implementing a Priority Queue ADT
  - Operations on a Heap
  - Building a Heap via Heapify
  - Analysis of Operations
  - Brief Introduction to *d*-Heaps

Learning Goals

- Define terminology about trees.
- Provide examples of appropriate applications for priority queues and heaps.
- Manipulate data in heaps.
- Describe and apply the Heapify algorithm, and analyze its complexity.

Rooted Trees and Some Applications

- Family Trees
- Organization Charts
- Classification Trees
  - What kind of flower is this?
  - Is this mushroom poisonous?
- File Directory Structure
  - Folders and Subfolders in Windows
  - Directories and Subdirectories in UNIX
- Non-Recursive Call Graphs
- Indexes in Database Systems

Tree Terminology: Examples

root:

leaf:

child of ______ :

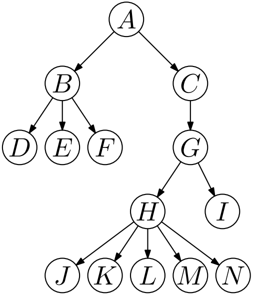
parent of ______ :

sibling:

ancestor of ______ :

descendent of ______ :

subtree of ______ :



Tree Terminology Reference

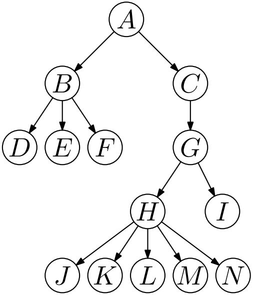
root: the single node with no parent

leaf: a node with no children

child: a node pointed to by me

parent: the node that points to me

sibling: another child of my parent



ancestor: my parent or my parent's ancestor

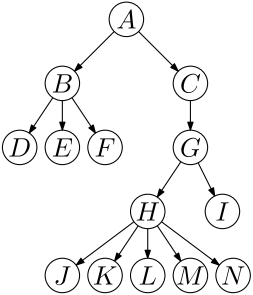
descendent: my child or my child's descendent

subtree: a node <u>and</u> its descendents

More Tree Terminology

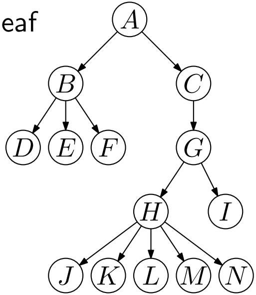
depth: number of edges on path from root to node

depth of H?



More Tree Terminology

height: number of edges on longest path from a given node to its furthest descendent; or, when speaking of the whole tree: number of edges on longest path from root to leaf



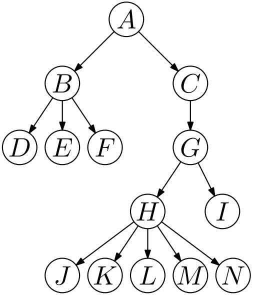
height of tree?

height of G?

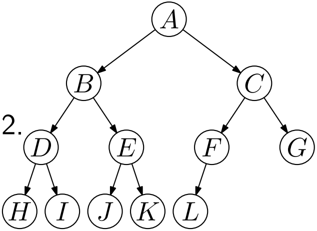
More Tree Terminology

(downward) degree: number of children of a given node

degree of B?



One More Tree-Terminology Slide



binary: Each node has degree at most 2.

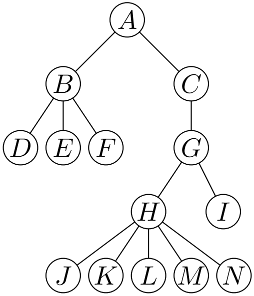
d-**ary:** The degree is at most *d*.

full: Each internal (non-leaf) node has the maximum number of children (2 in the case of a binary tree).

complete: It has as many nodes as possible for its height (i.e., each row is filled in).

nearly complete: Each row, except possibly the last one, is filled in, and all nodes in the last row are as far left as possible. (Warning: Some authors like Koffman/Wolfgang call this a *complete* tree. We'll stick with *nearly complete*.)

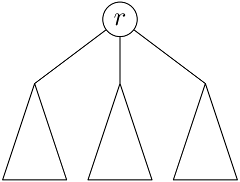
Example: Finding the Longest Undirected Path in a Tree



Does such a path always include the root?

Longest Path

An algorithm to find the longest *undirected* path in a tree:

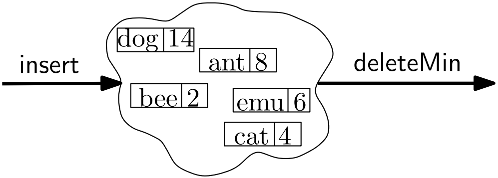


Back to Queues

- Applications
  - Ordering jobs/processes on a CPU
  - Simulating events
  - Picking the next search site
- But we don't necessarily want FIFO. You can choose your order, according to some carefully thought-out priority. Maybe:
  - *Shorter* jobs should go first.
  - *Earliest* (simulated time) events should go first.
  - *Most promising* sites should be searched first.

Priority Queue ADT

- Priority Queue Operations
  - create
  - destroy
  - insert
  - deleteMin
  - is_empty



- Priority Queue Property (in a minimum priority queue): For two elements in the queue, x and y, if x has a lower priority value than y, x will be deleted before y when performing a deleteMin operation.

Applications of a Priority Queue

- Hold jobs for a printer in order of length.
- Store packets on network routers in order of urgency.
- Simulate events.
- Select symbols for compression.
- Sort numbers.
- Anything *greedy*: In this case, an algorithm makes the "locally best choice" (not necessarily the overall best choice) at each step.

Priority Queue Data Structures

- Unsorted List
  - insert time:
  - deleteMin time:
- Sorted List
  - insert time:
  - deleteMin time:

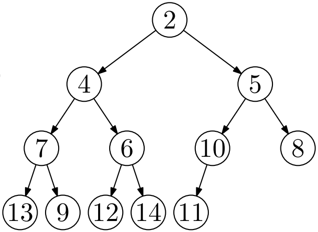
Binary Heap Priority Queue Data Structure

Heap-Order Property: parent's key $\leq$ children's key (we often call this a *minimum* heap)

- minimum is always at the top

Structure Property: "nearly complete tree"

- depth is always O(lg n)
- next open location is always known

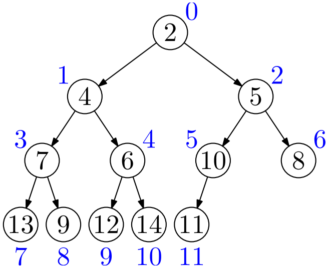


WARNING: This has no similarity to the memory "heap" we talk about when using C++'s `new` operator.

Nifty Storage Trick

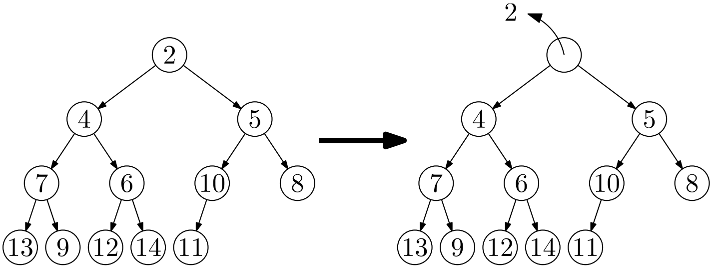
Navigation using indices:

- left_child(i) $=$
- right_child(i) $=$
- parent(i) $=$
- root $=$
- next free position $=$



0	1	2	3	4	5	6	7	8	9	10	11	12
2	4	5	7	6	10	8	13	9	12	14	11	

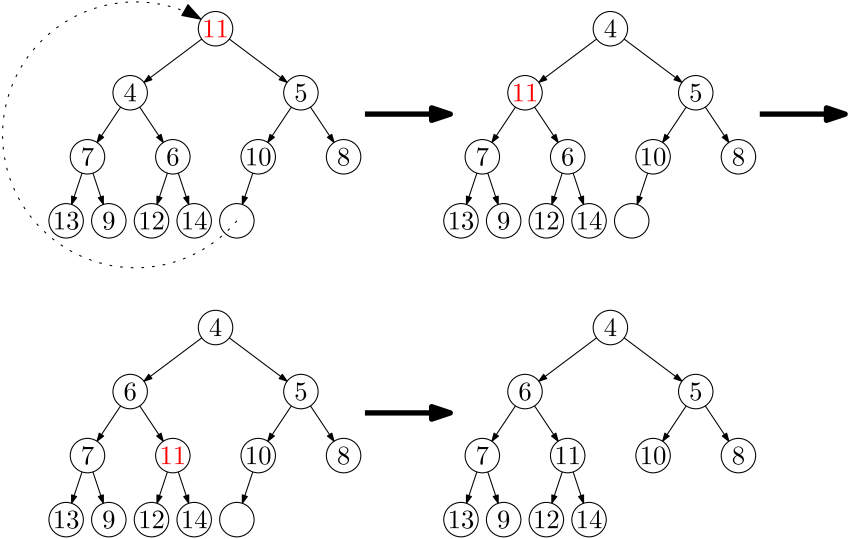
deleteMin



Invariants violated! It's no longer a "nearly complete" binary tree.

Swap (Heapify) Down

Move last element to the root, and then swap it down to its proper position.



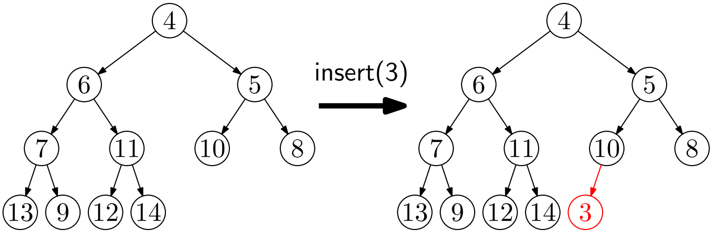
deleteMin Code

```
int deleteMin() {
  assert(!isEmpty());
  int returnVal = Heap[0];
  Heap[0] = Heap[n-1];
  n--;
  swapDown(0);
  return returnVal;
}
```

Runtime:

```
void swapDown(int i) {
  int s = i;
  int left = i * 2 + 1;
  int right = left + 1;
  if( left < n &&
      Heap[left] < Heap[s] )
    s = left;
  if( right < n &&
      Heap[right] < Heap[s] )
    s = right;
  if( s != i ) {
    int tmp = Heap[i];
    Heap[i] = Heap[s];
    Heap[s] = tmp;
    swapDown(s);
  }
}
```

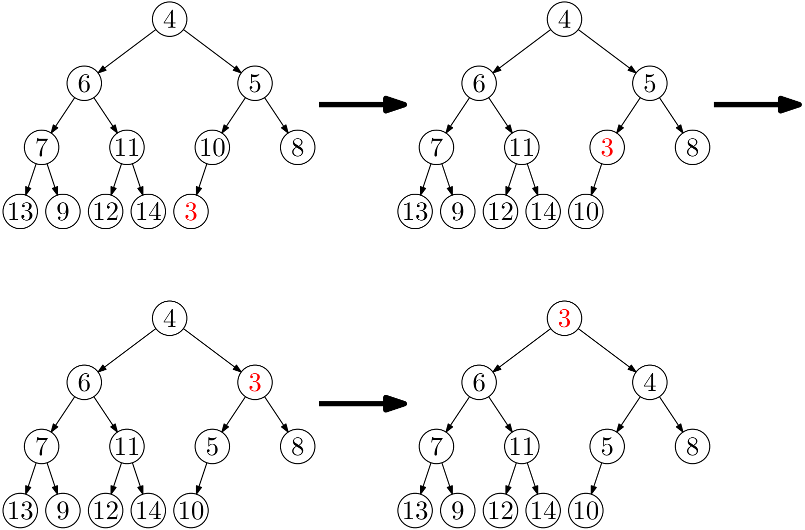
Inserting a New Node



Invariant violated! Child has smaller key than parent.

Swap (Heapify) Up

Begin by putting the new element last, then swap it up to its proper position.



insert Code

```
void insert(int x) {
  assert(!isFull());
  Heap[n] = x;
  n++;
  swapUp(n-1);
}
```

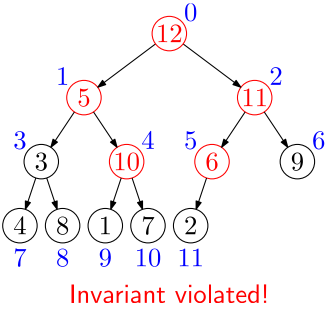
```
void swapUp(int i) {
  if( i == 0 ) return;
  int p = (i - 1)/2;
  if( Heap[i] < Heap[p] ) {
    int tmp = Heap[i];
    Heap[i] = Heap[p];
    Heap[p] = tmp;
    swapUp(p);
  }
}
```

Runtime:

Heapify: Build a Heap from an Array

1. Start with the input array.

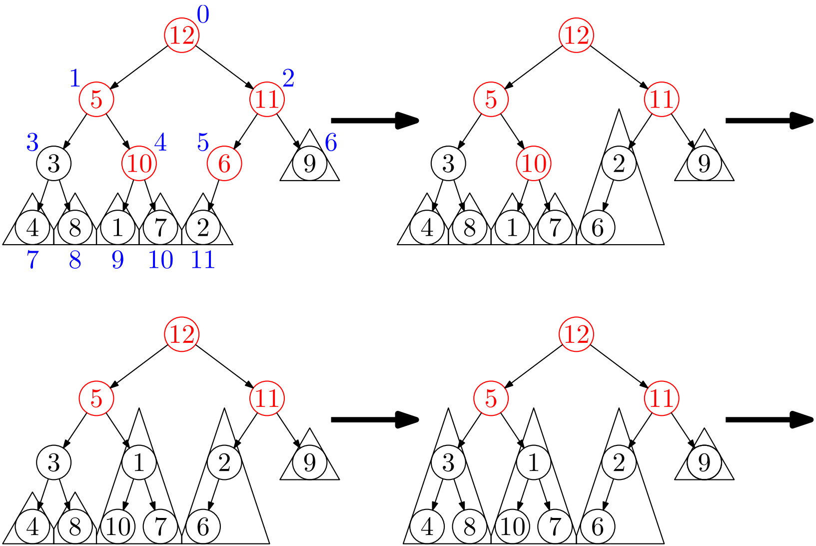
12	5	11	3	10	6	9	4	8	1	7	2



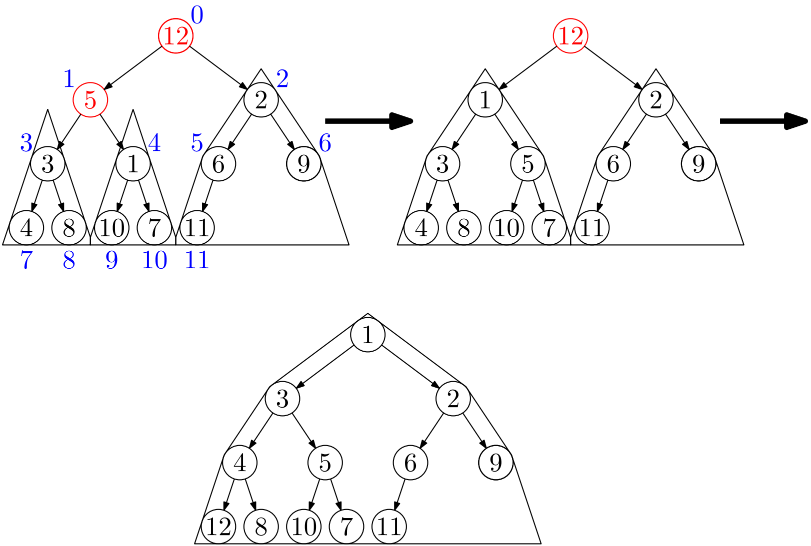
2. Fix the heap-order property, starting from the bottom, and going up. Use `swapDown`.

```
for( i = n/2 - 1; i >= 0; i-- )
     swapDown(i);
```

Heapify Example...

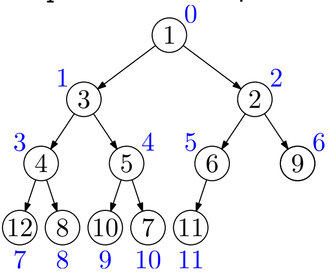


Heapify Example



Heapify Runtime

`swapDown` on a heap of height h takes at most ______________ steps.

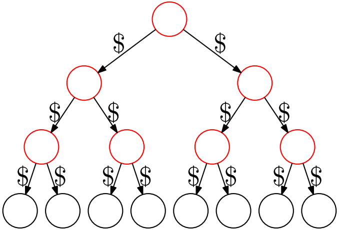


Let H be the height of the heap.

`swapDown` is called	once	on heap of height	H
	≤ 2 times	on heap of height	$H-1$
	≤ 4 times	on heap of height	$H-2$
	$\vdots$		
	$\leq 2^{H-1}$ times	on heap of height	1

Total # steps $\leq \sum_{h=1}^{H} h2^{H-h} = 2^H \sum_{h=1}^{H} h/2^h \leq 2^{H+1} = O(n)$

Heapify Runtime: Charging Scheme



Possible violations. How much time to fix them?

Place a dollar on each edge of the heap. One dollar pays for one step of `swapDown`. By induction, we can show that when `swapDown` is called on a node v, both children of v have a path (the rightmost path) to a leaf that is uncharged. The edges on the left child's rightmost path plus the edge to the left child pay for the steps of `swapDown` at v. The edges on the right child's rightmost path plus the edge to the right child form the uncharged path available to the parent of v.

Thinking about Binary Heaps

Observations

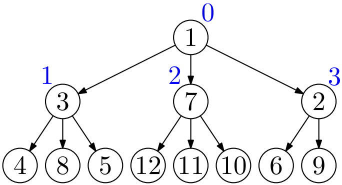
- Finding a child/parent index is a multiply/divide by two operation.
- Both deleteMin and the subsequent insert might access far-apart array locations.
- deleteMin accesses all children of visited nodes.
- insert accesses only the parent of visited nodes.
- insert is at least as common as deleteMin.

Realities

- Division and multiplication by powers of two are fast.
- Far-apart array accesses can ruin cache performance.
- With large datasets, disk I/O dominates CPU time.

Solution: d-Heaps

These are complete d-ary trees (representable by an array) with a heap-order property.



1	3	7	2	4	8	5	12	11	10	6	9
0	1	2	3	4	5	6	7	8	9	10	11

Good choices for d:

- fit one set of children on a memory page/disk block
- fit one set of children in a cache line
- optimize performance based on ratio of inserts/deleteMins
- make d a power of two for efficiency

d-Heap Navigation

- jth-child(i) $=$
- parent(i) $=$
- root $=$
- next free position $=$