Unit #7: B$^+$-Trees

CPSC 221: Algorithms and Data Structures

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Unit Outline

- Minimizing disk I/Os
- B^+-Tree properties
- Implementing B^+-Tree insert and delete
- Some final thoughts on B^+-Trees

Learning Goals

- Describe the structure, navigation and time complexity of a B^+-Tree.
- Insert and delete keys from a B^+-Tree.
- Relate M, L, the number of nodes, and the height of a B^+-Tree.
- Compare and contrast B^+-Trees with other data structures.
- Justify why the number of I/Os becomes a more appropriate complexity measure (than the number of CPU operations) when dealing with large datasets and their indexing structures (e.g., B^+-Trees).
- Explain the difference between a B-Tree and a B^+-Tree

Memory Hierarchy

Why worry about the number of disk I/Os?

Size		**Access Time**
hundreds of bytes	CPU registers	<1 cycle
tens of kilobytes	Cache memory L1	a few cycles
	L2	
megabytes	L3	tens of cycles
gigabytes	Main memory	hundreds of cycles
terabytes	Disk	millions of cycles

Time Cost: Processor to Disk

Processor

- Operates at a few GHz (gigahertz = billion cycles per second).
- Several instructions per cycle.
- Average time per instruction < 1ns (nanosecond $= 10^{-9}$ seconds).

Disk

- Seek time $\approx$ 10ms (ms = millisecond $= 10^{-3}$ seconds)
- (Solid State Drives have "seek time" $\approx$ 0.1ms.) 0.03ms

Result: 10 million instructions for each disk read!
Hold on... How long does it take to read a 1TB (terrabyte $= 10^{12}$ bytes) disk? 1TB $\times$ 10ms = 10 billion seconds $>$ 300 years?
What's wrong? Each disk read/write moves more than a byte.
Continuous disk access about the same speed as on SSD.

Memory Blocks

Each memory access to a slower level of the hierarchy fetches a block of data.

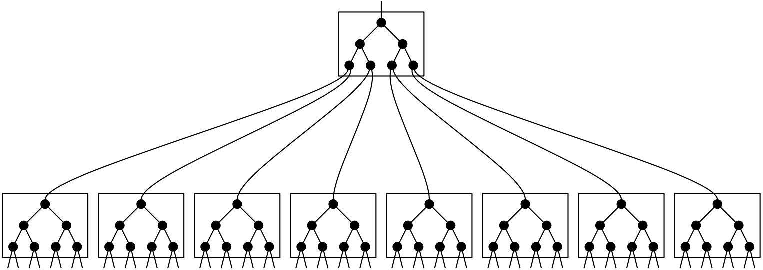
Block Size		Block Name
	CPU	
a few bytes	$\updownarrow$	word
	Cache	
10s bytes	$\updownarrow$	cache line
	Main memory	
a few kilobytes	$\updownarrow$	page
	Disk	

A block is the contents of consecutive memory locations.
So random access between levels of the hierarchy is very slow.

Chopping Trees into Blocks

Idea

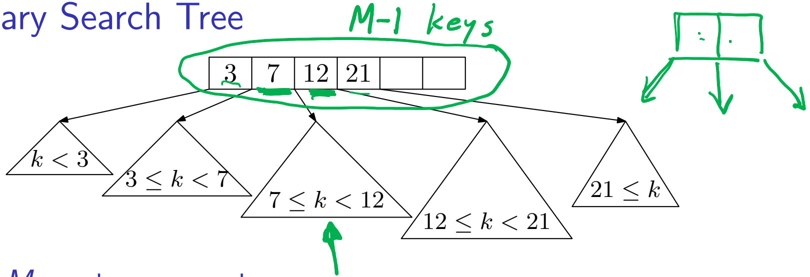
Store data for many adjacent nodes in consecutive memory locations.



Result

One memory block access provides keys to determine many (more than two) search directions.

M-ary Search Tree



M-ary tree property

- Each node has $\le M$ children

Result: Complete M-ary tree with n nodes has height $\Theta(\log_M n)$

Search tree property

- Each node has $\le M - 1$ search keys: $k_1 < k_2 < k_3 \ldots$
- All keys k in ith subtree obey $k_i \le k < k_{i+1}$ for $i = 0, 1, \ldots$.

worst case

Disk I/O's (runtime) for `find`:

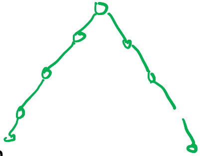
items = n

a) $\Theta(\log_M n)$

b) $\Theta(\log n)$

c) $\Theta(n)$

B$^+$-Trees



B$^+$-Trees of order M are specialized M-ary search trees:

balance property

- ALL leaves are at the same depth!
- Internal nodes have between $\lceil M/2 \rceil$ and M children
- Values are stored only at leaves. Search keys in internal nodes only direct traffic. B-Trees store (key, value) pairs at internal nodes.

$\equiv$ non leaf

- Leaves hold between $\lceil L/2 \rceil$ and L (key, value) pairs.
- The root is special. If internal, it has between 2 and M children. If a leaf, it holds at most L (key, value) pairs.

Result

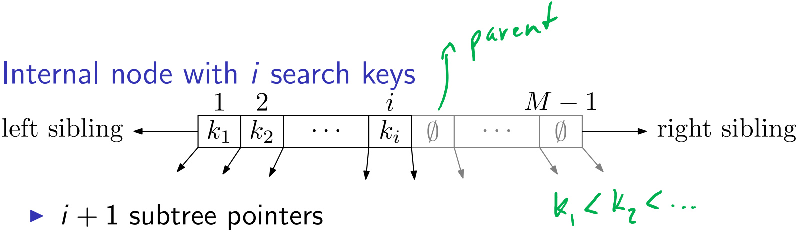
- Height is $\Theta(\log_M n)$ — If M is constant, this is $\Theta(\log n)$

M is big

- Insert, delete, find visit $\Theta(\log_M n)$ nodes
- M and L are chosen so that each (full) node fills one page of memory. Each node visit (disk I/O) retrieves about $M/2$ to M keys or $L/2$ to L (key, value) pairs at a time.

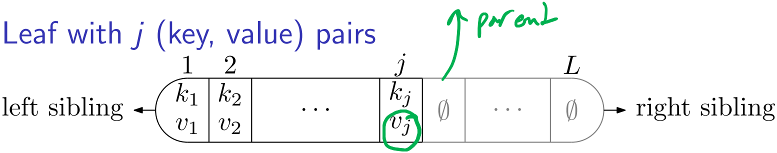
B^+-Tree Nodes

Internal node with i search keys



- $i+1$ subtree pointers
- parent and left & right sibling pointers

Leaf with j (key, value) pairs



- parent and left & right sibling pointers
- values may be pointers to disk records

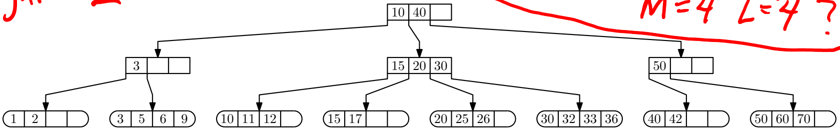
Each node may hold a different number of items.

Example B^+-Tree with $M = 4$ and $L = 4$

$2 \leq \#\text{children} \leq 4$

23 entries
height = 2

max # entries in B^+ tree of height 2 $M=4$ $L=4$?

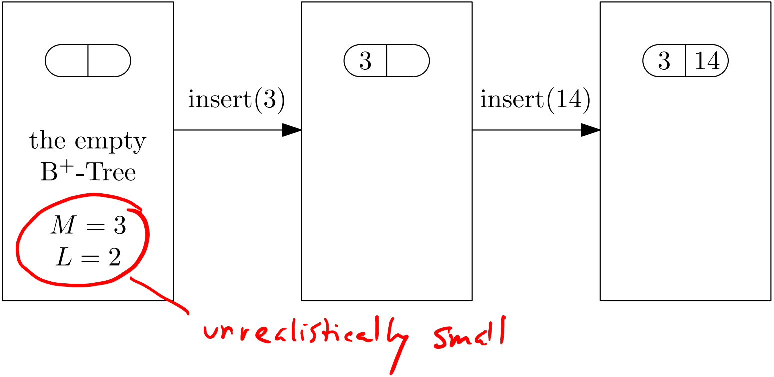


BST would have height at least? 2 3 4 5 6

Values in leaf nodes are not shown.

nodes in BST of height $h \leq 2^{h+1} - 1$

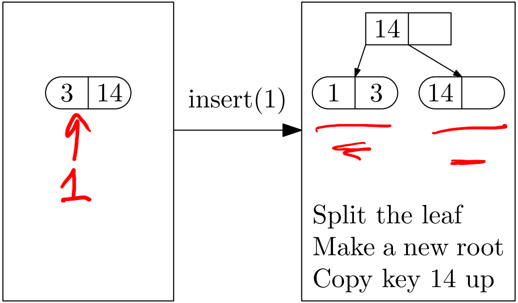
Making a B$^+$-Tree



The root is a leaf.

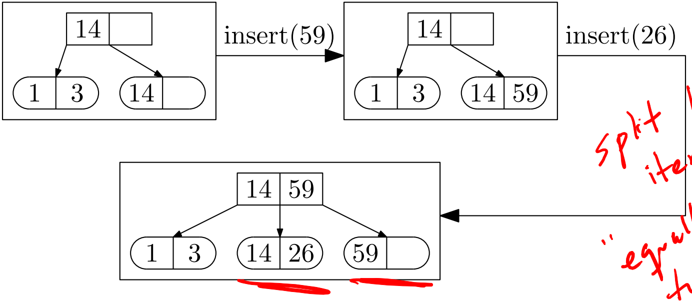
What happens when we now insert(1)?

Splitting the Root



Too many keys for one leaf!
So, make a new leaf and create a parent (the new root) for both.
Why are there duplicate 14 keys? direct traffic

Splitting a Leaf

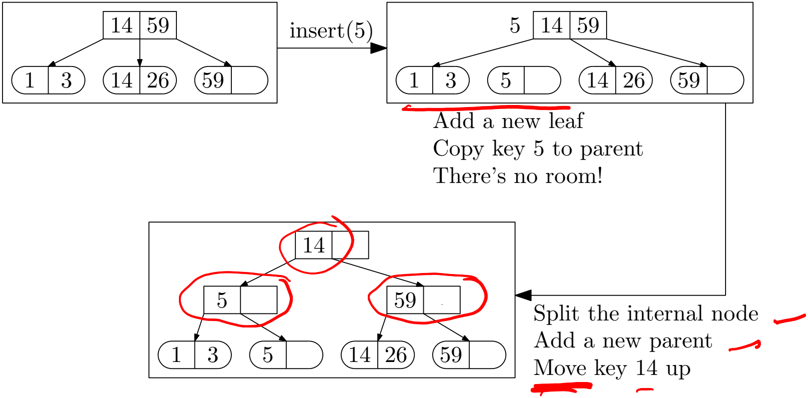


split leaf items "equally" between two leaves

insert(26) causes too many keys for the (14 | 59) leaf.

So, make a new leaf and **copy** the "middle" key (the smallest key in the new leaf holding the larger keys) up to the common parent.

Propagating Splits



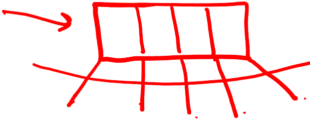
insert(5) causes too many keys for (1 | 3) leaf.

Copy up key 5 causes too many keys for [14 | 59] node.

So, make a new internal node and **move up** the middle key.

Insertion Algorithm

M-1 keys



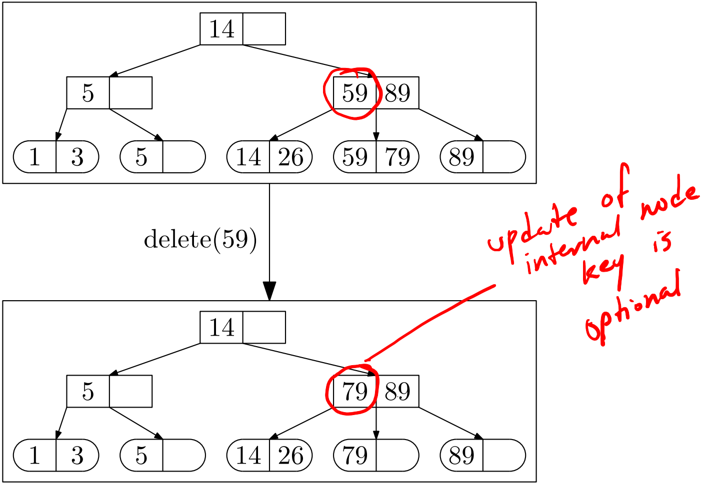
M=5

1. Insert (key, value) pair in its leaf.
2. If the leaf now has $L+1$ pairs: // overflow
   - Split the leaf into two leaves:
     - Original holds the $\lceil (L+1)/2 \rceil$ small key pairs.
     - New one holds the $\lfloor (L+1)/2 \rfloor$ large key pairs.

     overful leaf becomes 2 half-full leaves
   - **Copy** smallest key in new leaf (the middle key) up to parent.
3. If an internal node now has M keys: // overflow
   - Split the node into two nodes:
     - Original holds the $\lceil (M-1)/2 \rceil$ small keys.
     - New one holds the $\lfloor (M-1)/2 \rfloor$ large keys.

     overful node becomes 2 half-full nodes
   - If root, hang the new nodes under a new root. Done.
   - **Move** the remaining middle key up to parent & Goto 3.

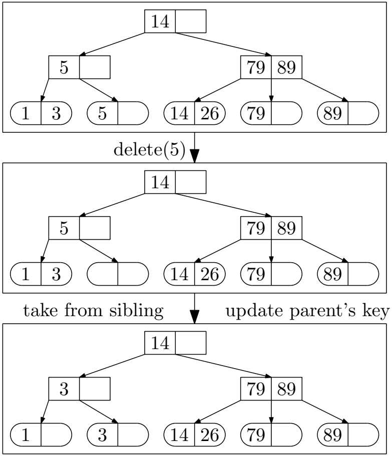
Delete



14
5
59 89
1 3
5
14 26
59 79
89
delete(59)
14
5
79 89
1 3
5
14 26
79
89
update of internal node key is optional

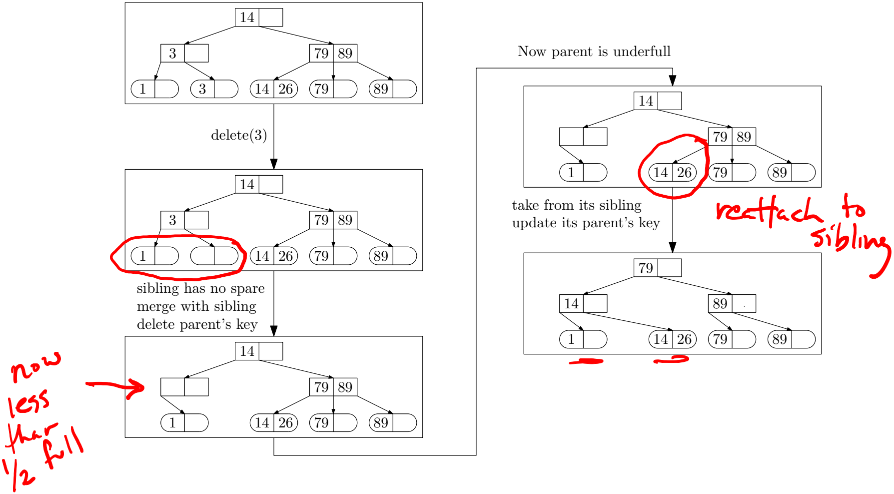
Delete: Take from a sibling

less than half full is unacceptable.



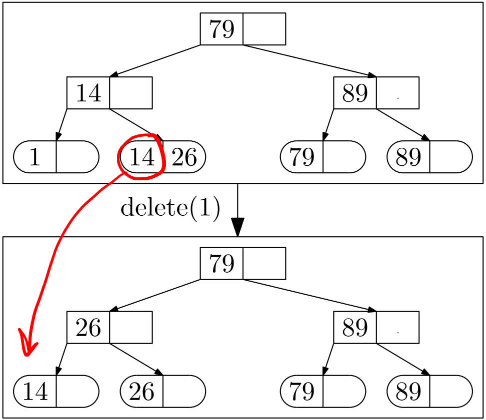
Take 3 from (1 | 3). It has enough items that it can spare one.
Update parent's search key. ← not optional

Delete: Merge

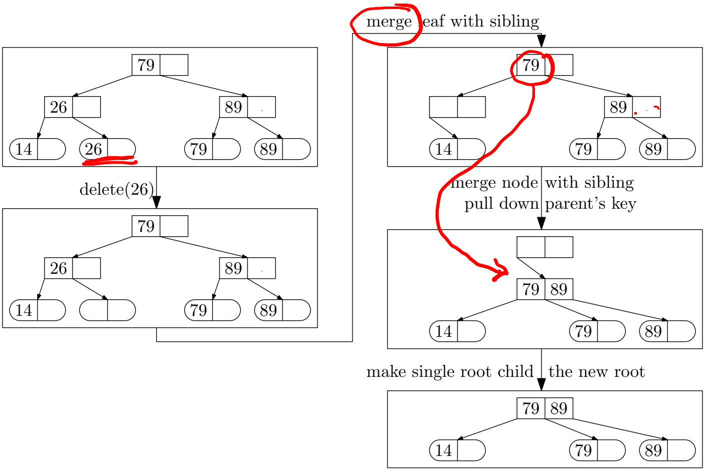


WARNING: A leaf is underfull if it holds fewer than $\lceil L/2 \rceil$ items.
For $L > 2$, an underfull leaf is not empty!

Delete: Take from a sibling



Delete: Killing the root



The root only gets deleted when it has just one subtree (no matter how big M is).

Deletion Algorithm

1. Remove (key, value) pair from its leaf.
2. If the leaf now has $\lceil L/2 \rceil - 1$ items, // underflow
   - If a sibling has a spare item then take it (smallest from right sibling or largest from left sibling) & update parent's key
   - Else merge with a sibling & **delete** parent's key
3. If internal non-root node now has $\lceil M/2 \rceil - 2$ keys, // underflow
   of internal node
   - If a sibling has a spare child then take it (leftmost from right sibling or rightmost from left sibling) & update parent's key
   - Else merge with a sibling & **pull down** parent's key & goto 3.
4. If the root now has only one child, make that child the new root.

Note: Merge never creates a node with too many items. Why?

taking from sibling didn't work so sibling is half full.

key that separated the siblings

Thinking about B$^+$-Trees

- Delete is fast if leaf doesn't underflow or we can take from a sibling. Merging and propagation take more time.
- Insert is fast if leaf doesn't overflow. (Could we give to a sibling?) Splitting and propagation take more time.
- Propagation is rare if M and L are large (Why?)
- Repeated insertions and deletion can cause thrashing
- If $M = L = 128$, then a B$^+$-Tree of height 4 will store at least 30,000,000 items
- Range queries (i.e., `findBetween(key1, key2)`) are fast because of sibling pointers.

2

64

64

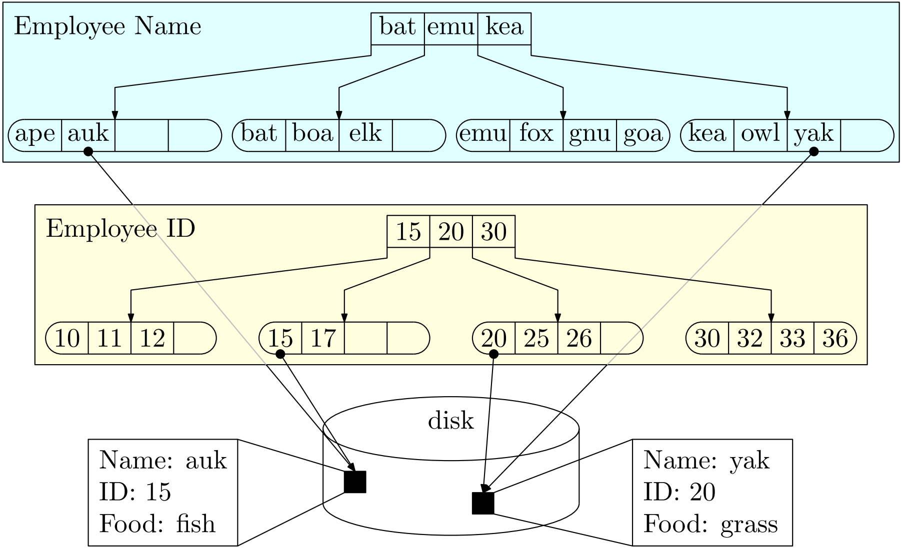
64 ×64

64

$2 \times 64^4 = 33554432$

B$^+$-Trees in practice

Multiple B$^+$-Trees can **index** the same data records.



A Tree by Any Other Name...

- B-Trees with $M = 3$ are called 2-3 trees
- B-Trees with $M = 4$ are called 2-3-4 trees

Why would we ever use these?