Unit #7: B$^+$-Trees

CPSC 221: Algorithms and Data Structures

Will Evans and Jan Manuch

2016W1

Unit Outline

- Minimizing disk I/Os
- B^+-Tree properties
- Implementing B^+-Tree insert and delete
- Some final thoughts on B^+-Trees

Learning Goals

- Describe the structure, navigation and time complexity of a B^+-Tree.
- Insert and delete keys from a B^+-Tree.
- Relate M, L, the number of nodes, and the height of a B^+-Tree.
- Compare and contrast B^+-Trees with other data structures.
- Justify why the number of I/Os becomes a more appropriate complexity measure (than the number of CPU operations) when dealing with large datasets and their indexing structures (e.g., B^+-Trees).
- Explain the difference between a B-Tree and a B^+-Tree

Memory Hierarchy

Why worry about the number of disk I/Os?

Size		**Access Time**
hundreds of bytes	CPU registers	< 1 cycle
tens of kilobytes	Cache memory L1	a few cycles
	L2	
megabytes	L3	tens of cycles
gigabytes	Main memory	hundreds of cycles
terabytes	Disk	millions of cycles

Time Cost: Processor to Disk

Processor

- Operates at a few GHz (gigahertz $=$ billion cycles per second).
- Several instructions per cycle.
- Average time per instruction < 1ns (nanosecond $= 10^{-9}$ seconds).

Disk

- Seek time $\approx$ 10ms (ms $=$ millisecond $= 10^{-3}$ seconds)
- (Solid State Drives have “seek time” $\approx$ 0.1ms.) 0.03ms

Result: 10 million instructions for each disk read!

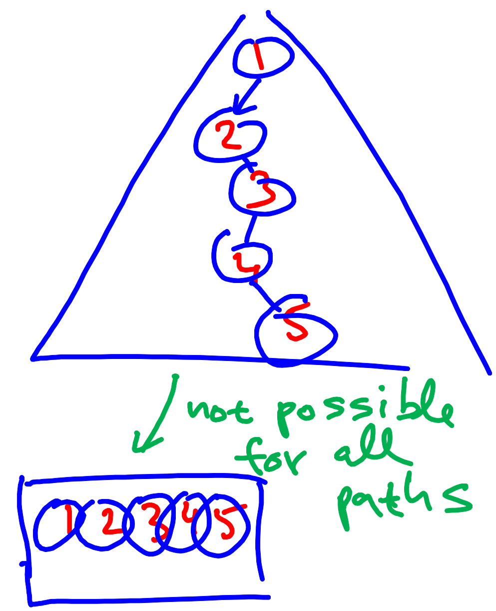
Hold on... How long does it take to read a 1TB (terrabyte $= 10^{12}$ bytes) disk? 1TB $\times$ 10ms $=$ 10 billion seconds $>$ 300 years?

What’s wrong? Each disk read/write moves more than a byte. Continuous disk access about the same speed as on SSD.

Memory Blocks

Each memory access to a slower level of the hierarchy fetches a block of data.

Block Size		**Block Name**
	CPU	
a few bytes	↕	word
	Cache	
10s bytes	↕	cache line
	Main memory	
a few kilobytes (4?)	↕	page
	Disk	

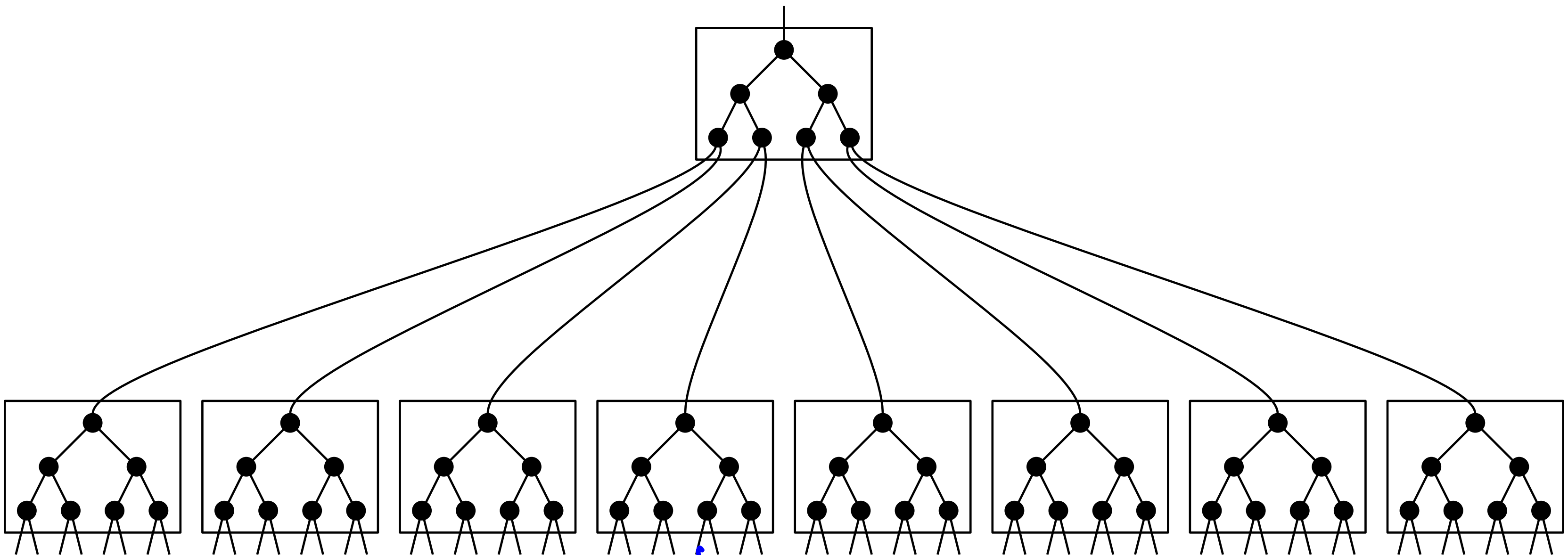


A block is the contents of consecutive memory locations.
So random access between levels of the hierarchy is very slow.

Chopping Trees into Blocks

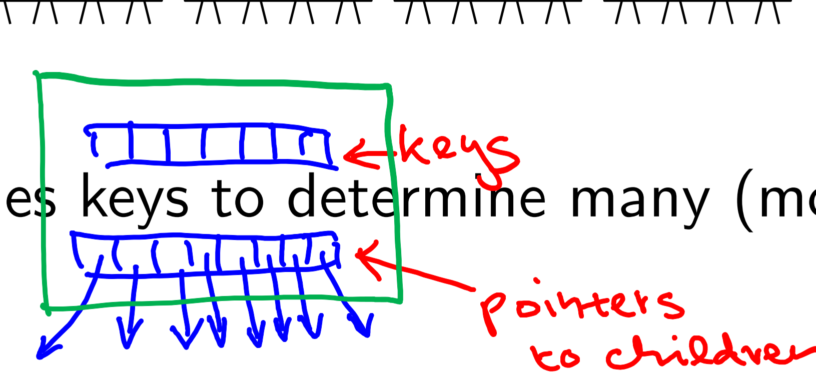
Idea

Store data for many adjacent nodes in consecutive memory locations.

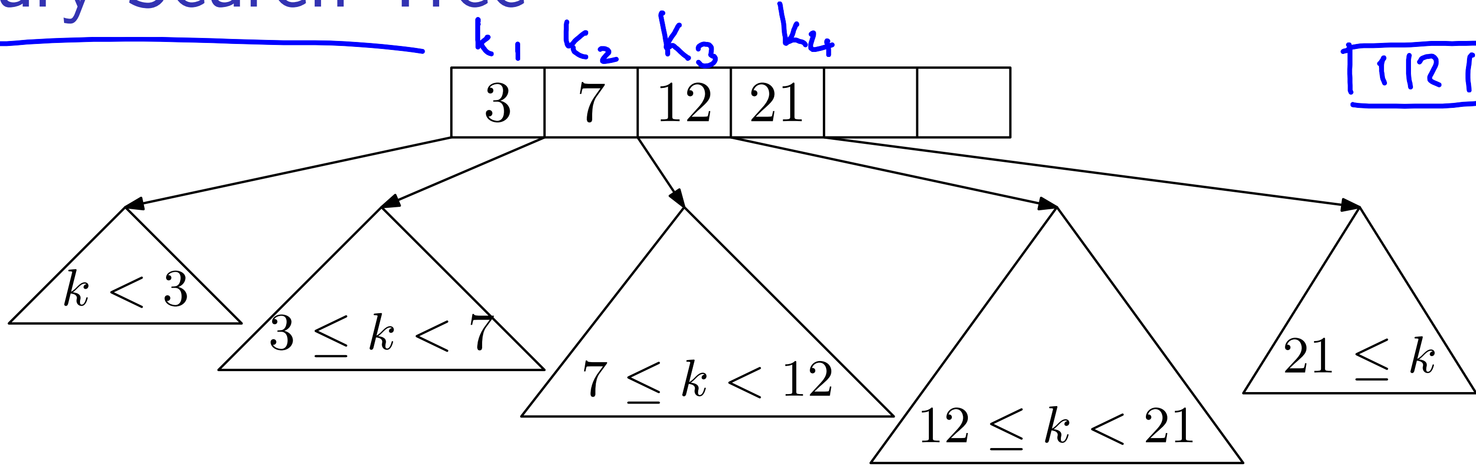


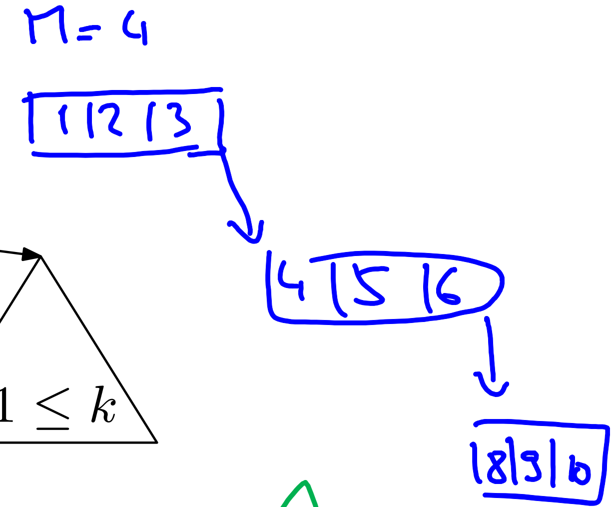
Result

One memory block access provides keys to determine many (more than two) search directions.



M-ary Search Tree





M-ary tree property

- Each node has $\leq M$ children

Result: Complete M-ary tree with n nodes has height $\Theta(\log_M n)$

Search tree property

- Each node has $\leq M - 1$ search keys: $k_1 < k_2 < k_3 \ldots$
- All keys k in ith subtree obey $k_i \leq k < k_{i+1}$ for $i = 0, 1, \ldots$.

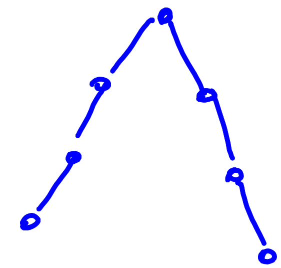
worst-case

Disk I/O's (runtime) for `find`:

1) $\Theta(\log n)$
2) $\Theta(\log_M n)$
3) $\Theta(n)$

we need to balance the tree!

B$^+$-Trees



B$^+$-Trees of order M are specialized M-ary search trees:

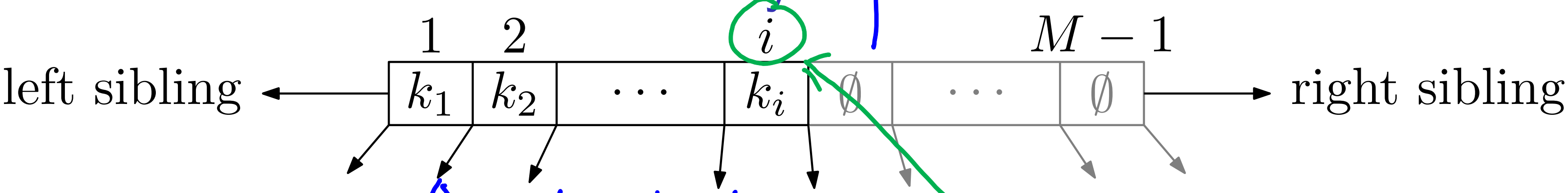
- ALL leaves are at the same depth!
- Internal nodes have between $\lceil M/2 \rceil$ and M children
- Values are stored only at leaves. Search keys in internal nodes only direct traffic. B-Trees store (key, value) pairs at internal nodes.
- Leaves hold between $\lceil L/2 \rceil$ and L (key, value) pairs.
- The root is special. If internal, it has between 2 and M children. If a leaf, it holds at most L (key, value) pairs.

Result

- Height is $\Theta(\log_M n)$
- Insert, delete, find visit $\Theta(\log_M n)$ nodes
- M and L are chosen so that each (full) node fills one page of memory. Each node visit (disk I/O) retrieves about $M/2$ to M keys or $L/2$ to L (key, value) pairs at a time.

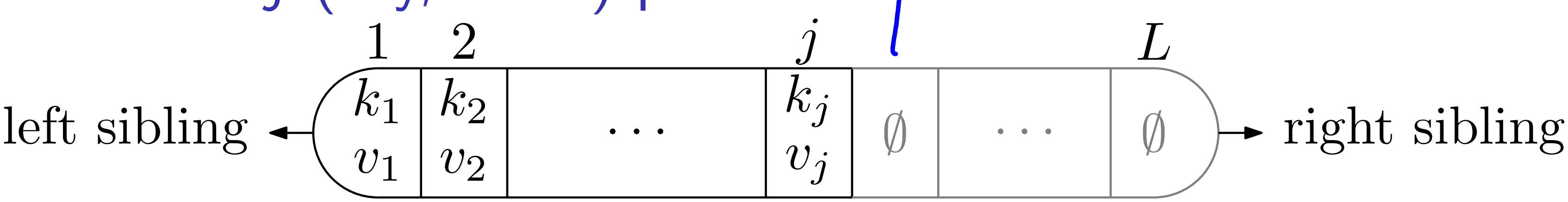
B^+-Tree Nodes

Internal node with i search keys



- $i+1$ subtree pointers
- parent and left & right sibling pointers

Leaf with j (key, value) pairs



- parent and left & right sibling pointers
- values may be pointers to disk records

Each node may hold a different number of items.

Example B^+-Tree with $M = 4$ and $L = 4$

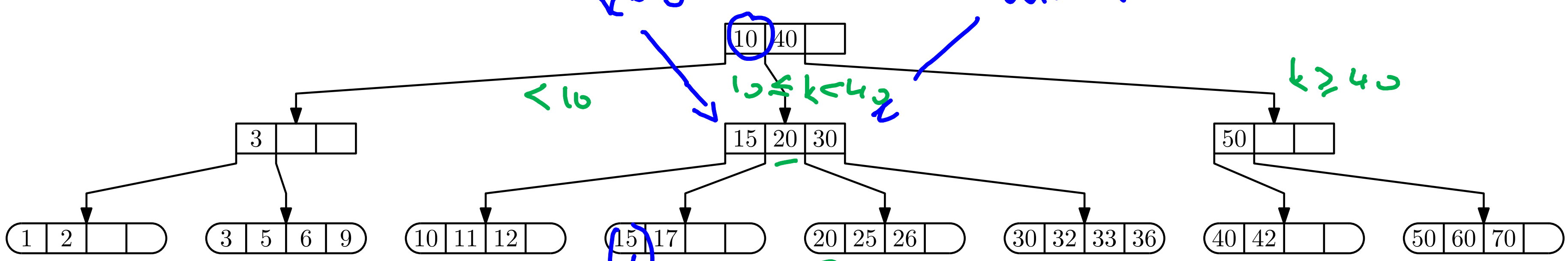
$2 \le$ # children ≤ 4

$1 \le$ # keys ≤ 3

$2 \le$ # entries ≤ 4

only key

direct traffic



height = 2

entries: 23

Values in leaf nodes are not shown.

Q. BST with 23 entries

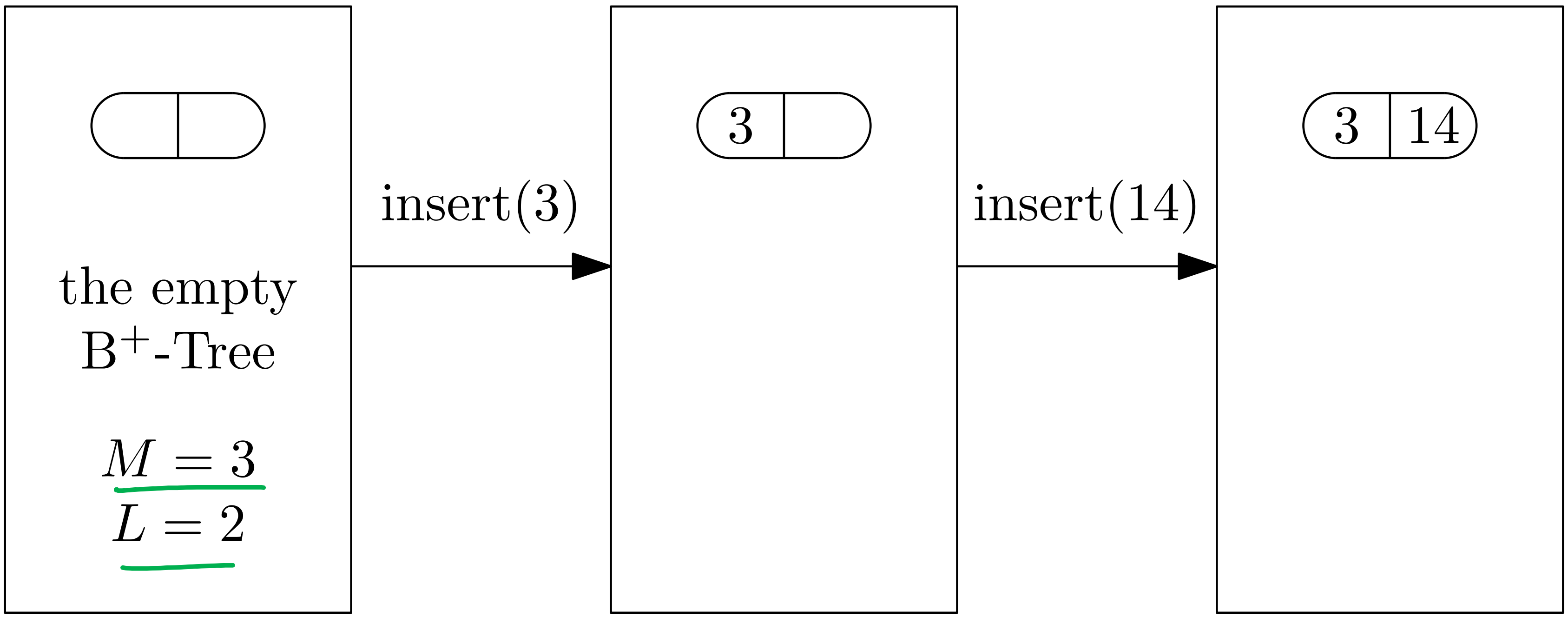
what is the smallest height?

$h = 4$

BST with height h

max # nodes: $2^{h+1} - 1$

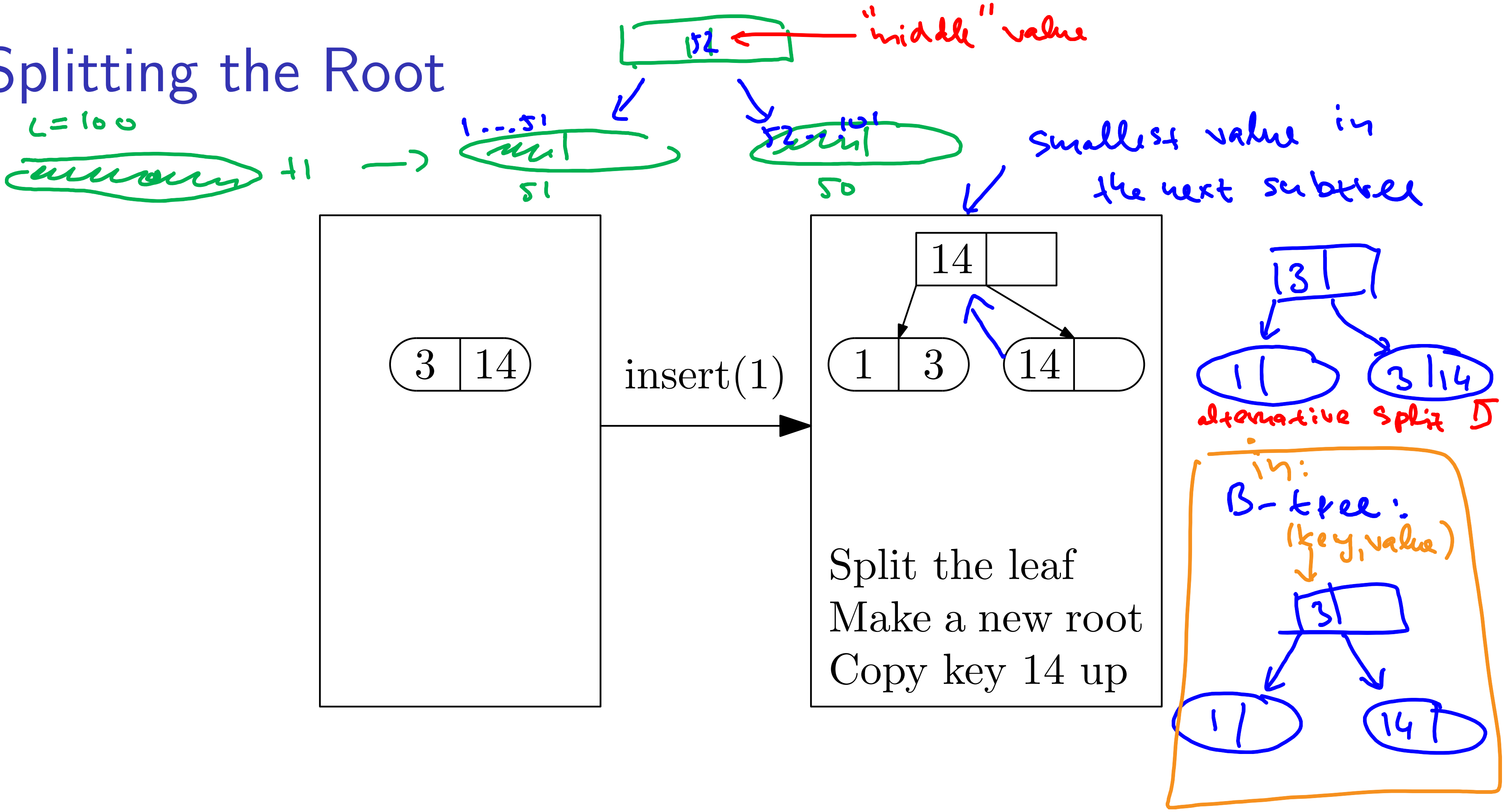
Making a B^+-Tree



The root is a leaf.

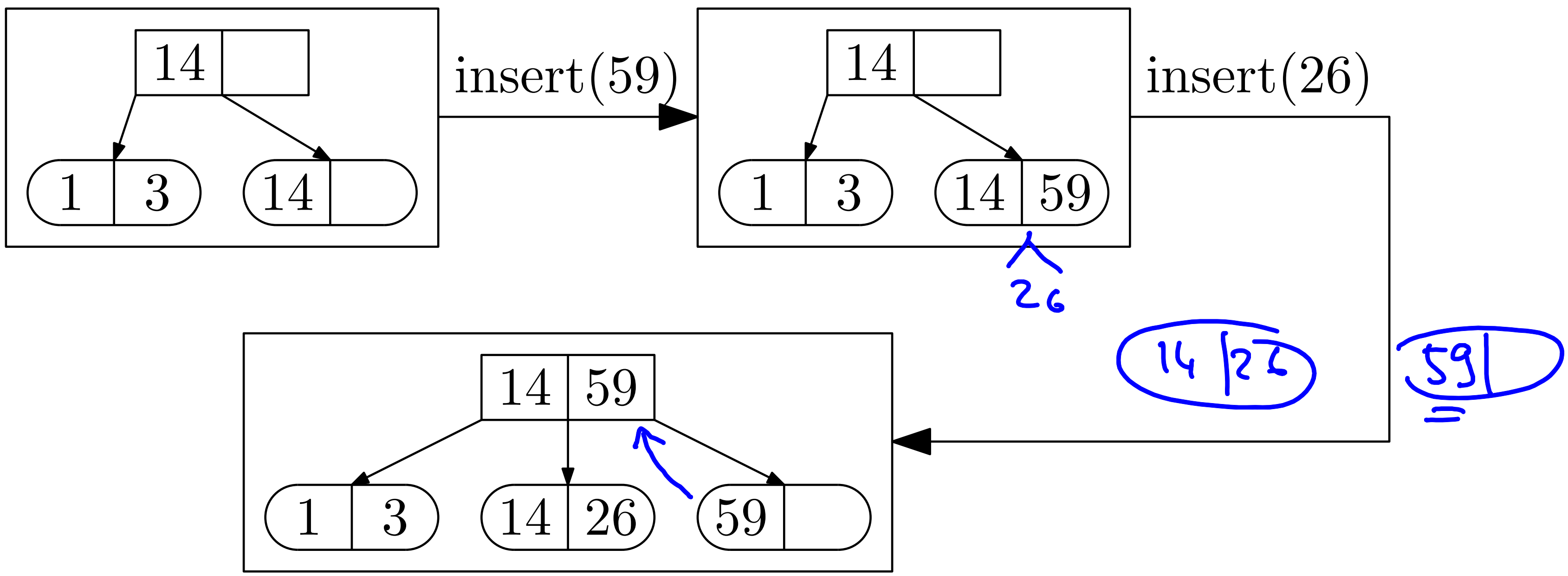
What happens when we now insert(1)?

Splitting the Root



Too many keys for one leaf!
So, make a new leaf and create a parent (the new root) for both.
Why are there duplicate 14 keys?

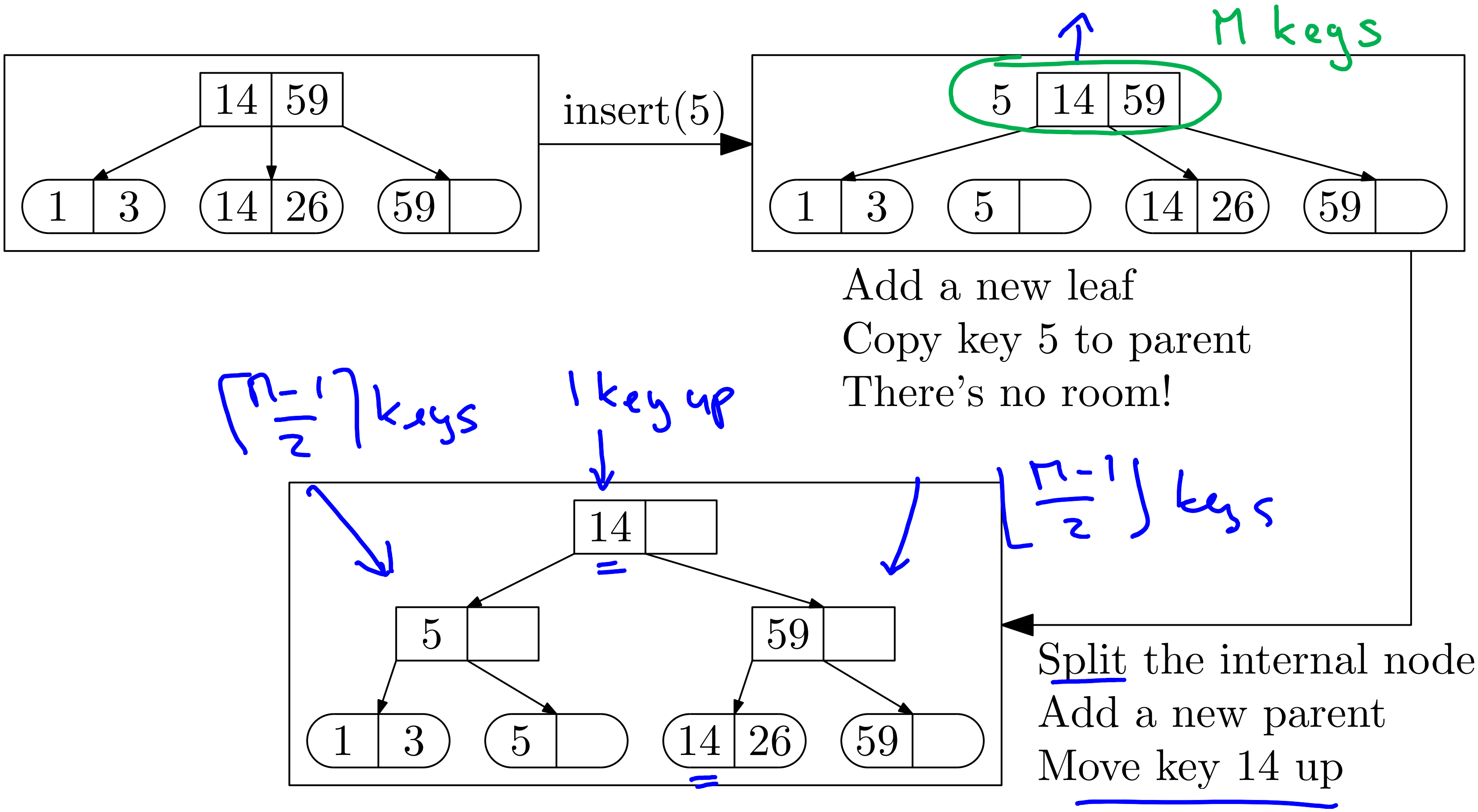
Splitting a Leaf



insert(26) causes too many keys for the (14 | 59) leaf.

So, make a new leaf and **copy** the middle key (the smallest key in the new leaf holding the larger keys) up to the common parent.

Propagating Splits



insert(5) causes too many keys for (1 | 3) leaf.

Copy up key 5 causes too many keys for [14 | 59] node.

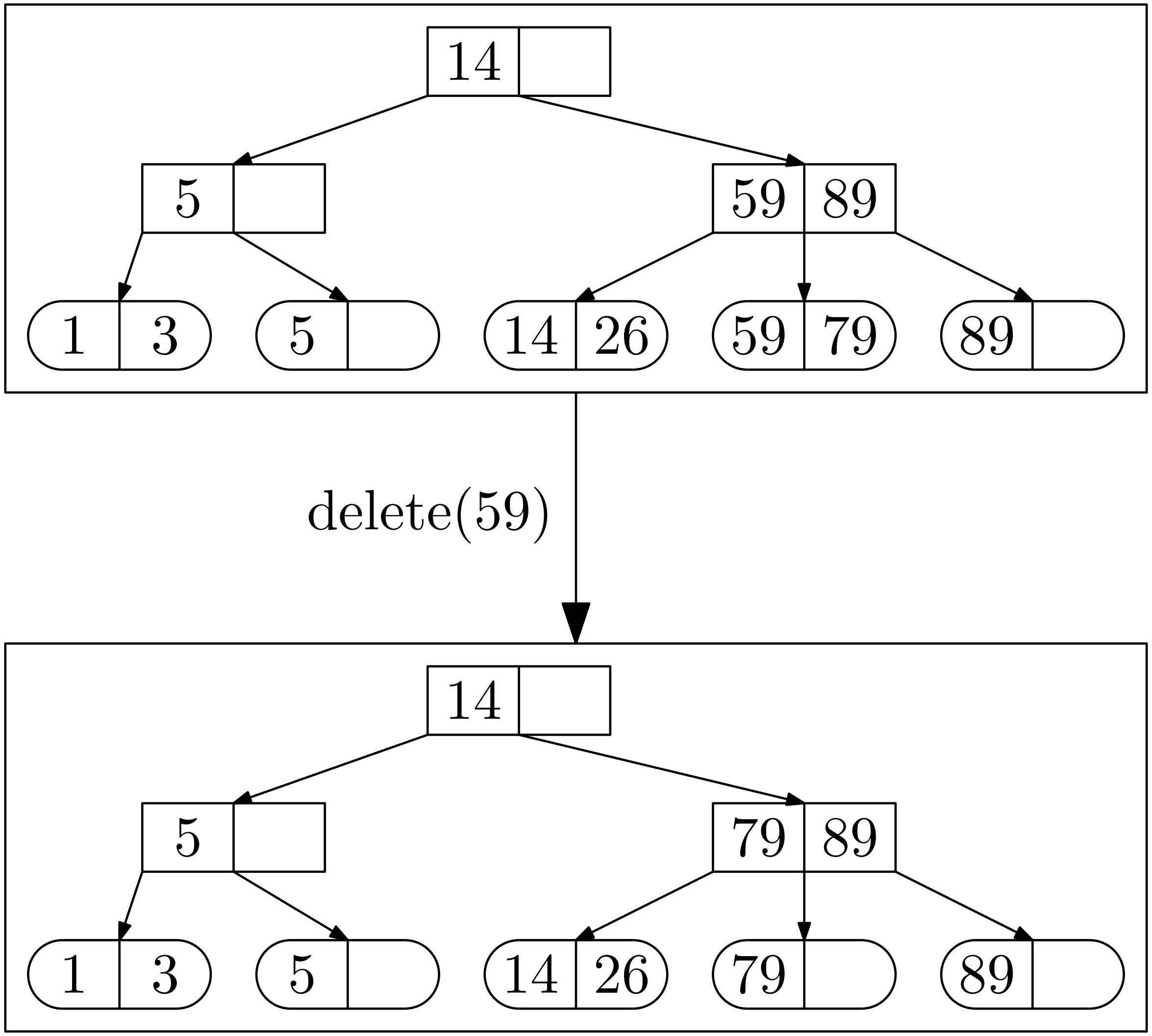
So, make a new internal node and **move up** the middle key.

Insertion Algorithm

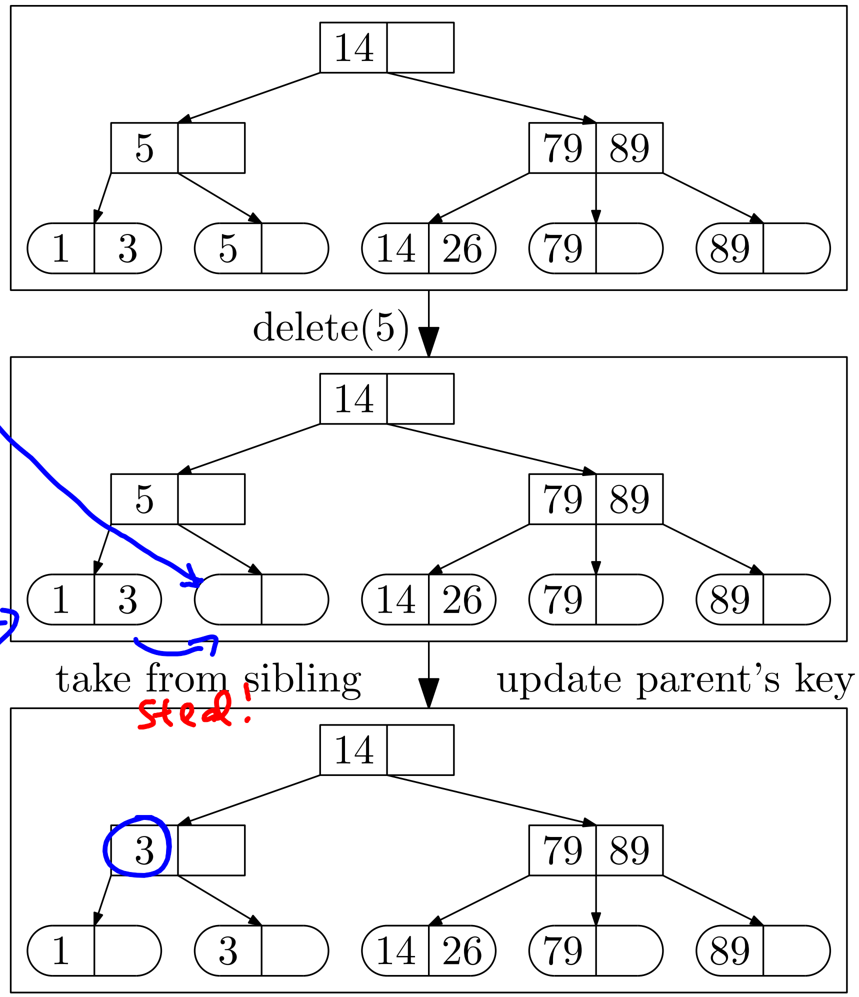
1. Insert (key, value) pair in its leaf.
2. If the leaf now has $L + 1$ pairs: // overflow
   - Split the leaf into two leaves: — overfull leaf becomes
     - Original holds the $\lceil (L+1)/2 \rceil$ small key pairs. } 2 half-full leaves
     - New one holds the $\lfloor (L+1)/2 \rfloor$ large key pairs. $\approx \lceil L/2 \rceil$
   - **Copy** smallest key in new leaf (the middle key) up to parent.
3. If an internal node now has M keys: // overflow
   - Split the node into two nodes: — overfull int.node becomes
     - Original holds the $\lceil (M-1)/2 \rceil$ small keys. } 2 half-full int.nodes
     - New one holds the $\lfloor (M-1)/2 \rfloor$ large keys. $\approx \lceil M/2 \rceil - 1$ .. so $\lceil M/2 \rceil$ children
   - If root, hang the new nodes under a new root. Done.
   - **Move** the remaining middle key up to parent & Goto 3.

Delete

remember:
valid # of keys:
1.. 2

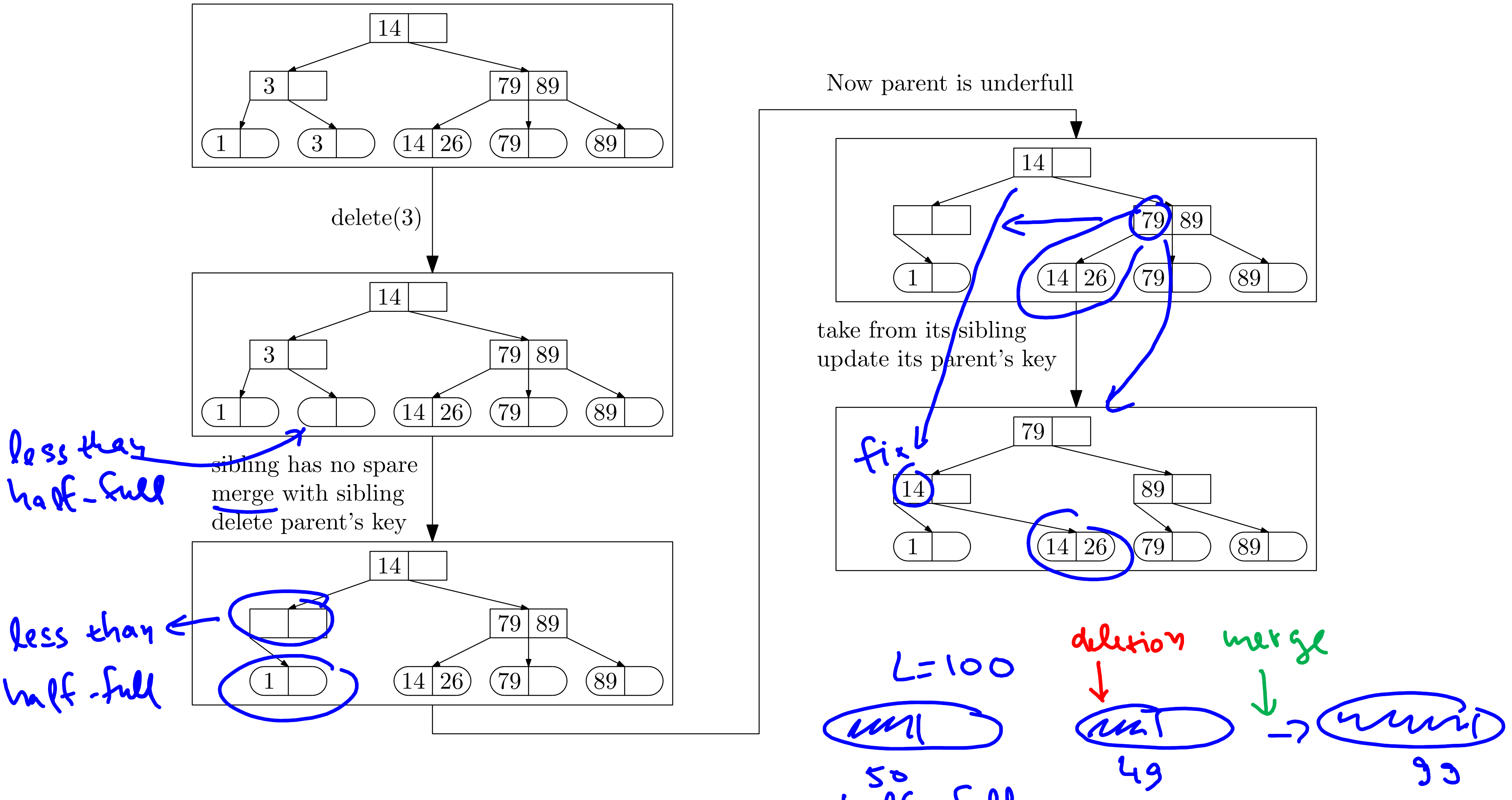


Delete: Take from a sibling



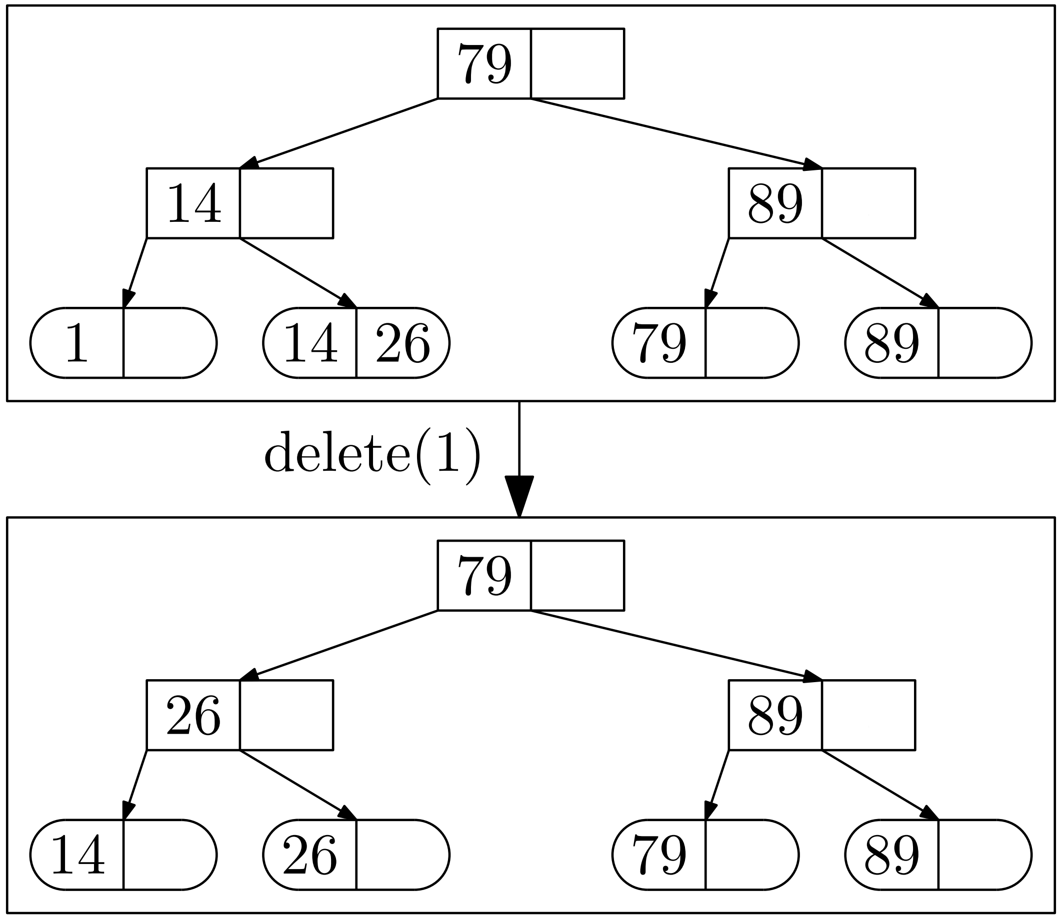
Take 3 from 1 3. It has enough items that it can spare one. Update parent's search key.

Delete: Merge

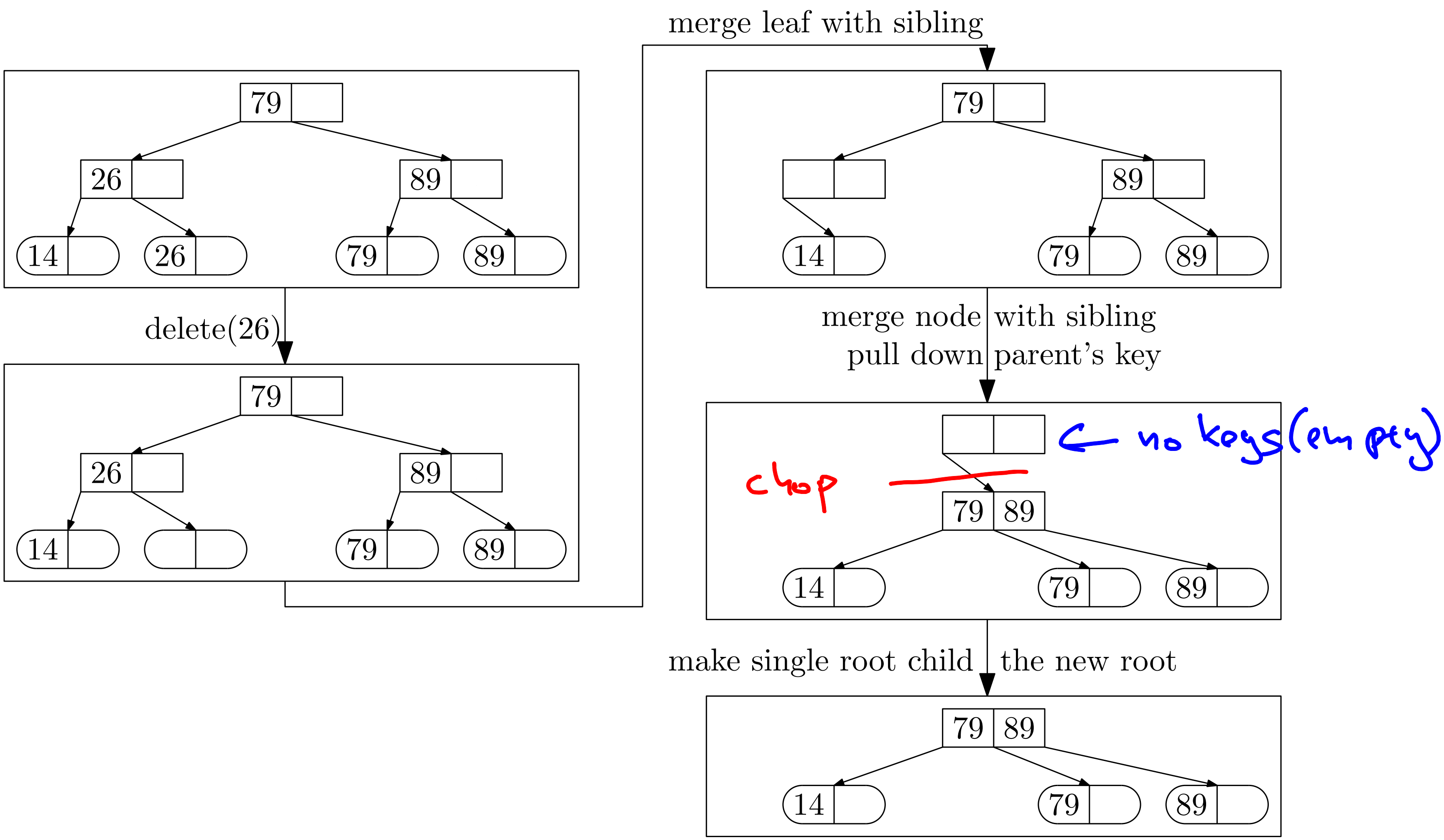


WARNING: A leaf is underfull if it holds fewer than $\lceil L/2 \rceil$ items.
For $L > 2$, an underfull leaf is not empty!

Delete: Take from a sibling



Delete: Killing the root



The root only gets deleted when it has just one subtree (no matter how big M is).

Deletion Algorithm

1. Remove (key, value) pair from its leaf.
2. If the leaf now has $\lceil L/2 \rceil - 1$ items, // underflow
   - If a sibling has a spare item then take it (smallest from right sibling or largest from left sibling) & update parent's key
   - Else merge with a sibling & **delete** parent's key
3. If internal non-root node now has $\lceil M/2 \rceil - 2$ keys, // underflow
   - If a sibling has a spare child then take it (leftmost from right sibling or rightmost from left sibling) & update parent's key
   - Else merge with a sibling & **pull down** parent's key & goto 3.
4. If the root now has only one child, make that child the new root.

   0 keys

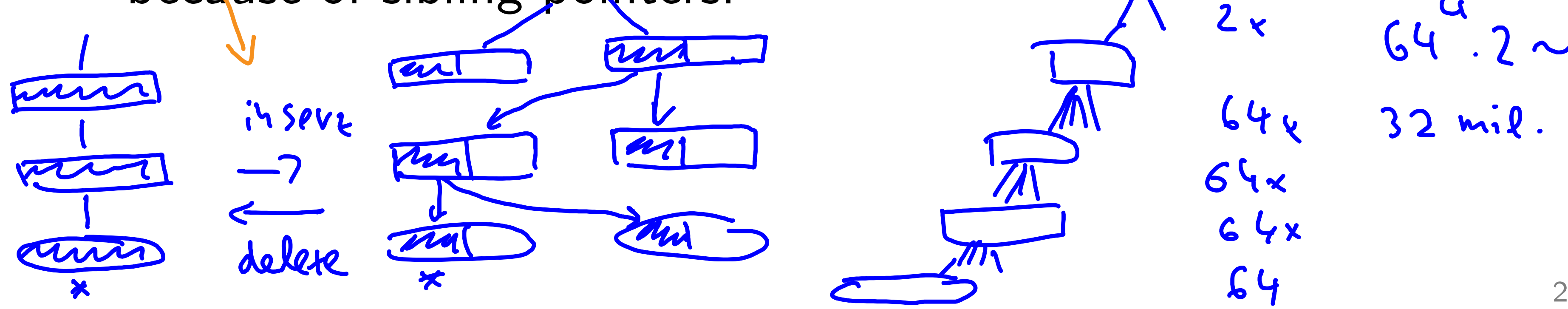
Note: Merge never creates a node with too many items. Why?

a leaf we have deleted from — its sibling

$$\left\lceil \frac{L}{2} \right\rceil - 1 + \left\lceil \frac{L}{2} \right\rceil = 2\left\lceil \frac{L}{2} \right\rceil - 1 = \begin{cases} L-1 & \text{if } L \text{ is even} \\ L & \text{if } L \text{ is odd} \end{cases}$$

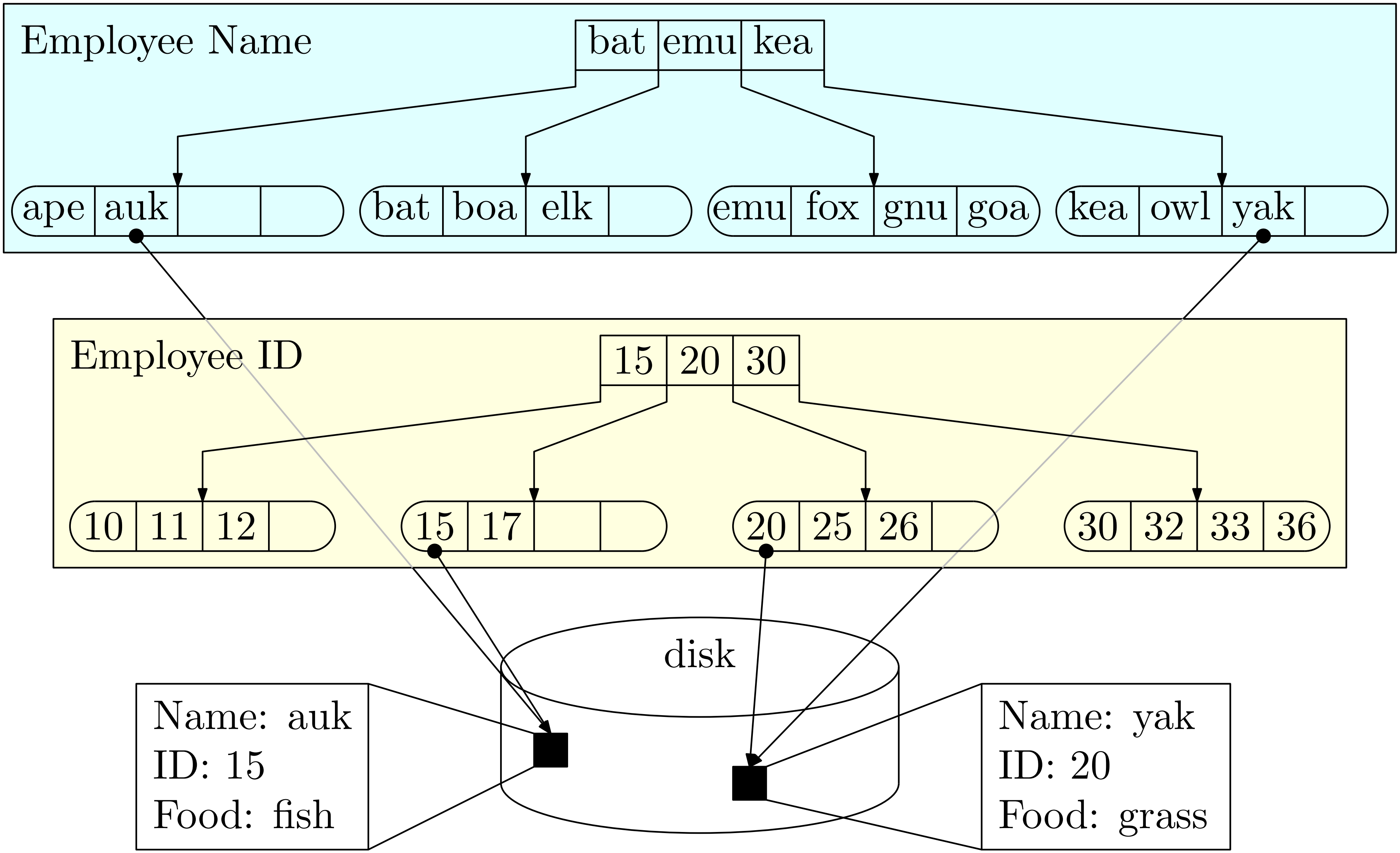
Thinking about B$^+$-Trees

- Delete is fast if leaf doesn't underflow or we can take from a sibling. Merging and propagation take more time.
- Insert is fast if leaf doesn't overflow. (Could we give to a sibling?) Splitting and propagation take more time.
- Propagation is rare if M and L are large (Why?)
- Repeated insertions and deletion can cause thrashing
- If $M = L = 128$, then a B$^+$-Tree of height 4 will store at least 30,000,000 items
- Range queries (i.e., `findBetween(key1, key2)`) are fast because of sibling pointers.



B^+-Trees in practice

Multiple B^+-Trees can **index** the same data records.



A Tree by Any Other Name...

- B-Trees with $M = 3$ are called 2-3 trees
- B-Trees with $M = 4$ are called 2-3-4 trees

Why would we ever use these?

- The height: $\Theta(\log n)$ ... as good as AVL
- other self-balancing trees: Red-Black Trees

  Splay Trees