

Unit #3: Recursion, Induction, and Loop Invariants

CPSC 221: Algorithms and Data Structures

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Unit Outline

- ▶ Thinking Recursively
- ▶ Recursion Examples
- ▶ Analyzing Recursion: Induction and Recurrences
- ▶ Analyzing Iteration: Loop Invariants
- ▶ How Computers Handle Recursion
 - ▶ Recursion and the Call Stack
 - ▶ Iteration and Explicit Stacks
 - ▶ Tail Recursion (KW text is wrong about this!)

Learning Goals

- ▶ Describe the relation between recursion and induction.
- ▶ Prove a program is correct using loop invariants and induction.
- ▶ Become more comfortable writing recursive algorithms.
- ▶ Convert between iterative and recursive algorithms.
- ▶ Describe how a computer implements recursion.
- ▶ Draw a recursion tree for a recursive algorithm.

Random Permutations (random permutation)

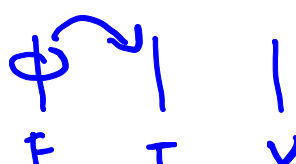
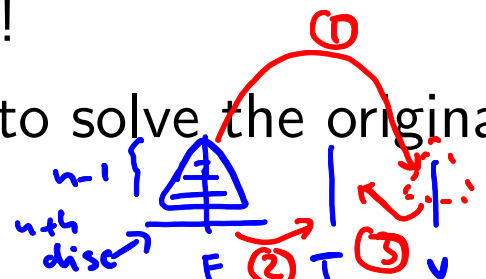
Problem: Permute a string so that every reordering of the string is equally likely.

One

Possible algorithm:

Pick a letter from the input string randomly
remove it from the input string
output to the output string
and repeat.

Thinking Recursively

1. DO NOT START WITH CODE. Instead, write the story of the problem, in natural language. *4 noi tower problem*
2. Define the problem: What should be done given a particular input?
3. Identify and solve the (usually simple) base case(s). *n=1*

4. Determine how to break the problem down into smaller problems of the same kind.
5. Call the function recursively to solve the smaller problems. Assume it works. Do not think about how!
6. Use the solutions to the smaller problems to solve the original problem.


Once you have all that, write the steps of your solution as comments and then fill in the code for each comment.

Random Permutations (random)

Problem: Permute a string so that every reordering of the string is equally likely.

Idea:

1. Pick a letter to be the first letter of the output. (Every letter should be equally likely.)
2. Pick the rest of the output to be a random permutation of the remaining string (without that letter).

It's slightly simpler if we pick a letter to be the **last** letter of the output.

Random Permutations (randomPermutation)

Problem: Permute a string so that every reordering of the string is equally likely.

```
// randomly permutes the first n characters of S
void permute(string &S, int n) {
    if( n > 1 ) {
        int i = rand() % n; // swap a random character of S
        char tmp = S[i]; // to the end of S.
        S[i] = S[n-1];
        S[n-1] = tmp;
        permute(S, n-1); // randomly permute S[0..n-2]
    }
}
```

parameter is passed by reference

passed by value:

(string S, int n) local var.

const string S that contains

a copy of original parameter

Swap
ith &
last
letter

(S, start+1, n-1)

permute (string &S, int start, int length)

in green: if we want to choose the first letter first

rand() % n returns an integer from $\{0, 1, \dots, n-1\}$ uniformly at random.

use reference: 1) you want modify the parameter
2) to avoid copying large objects

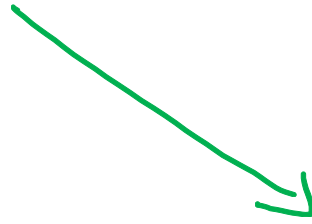
Induction and Recursion, Twins Separated at Birth?



Base case

Prove for some small value(s).

Inductive Step: Break a larger case down into smaller ones that we assume work (the Induction Hypothesis).



Base Case

Calculate for some small value(s).

Otherwise, break the problem down in terms of itself (smaller versions) and then call this function to solve the smaller versions, assuming it will work.

Proving a Recursive Algorithm is Correct

Just follow your code's lead and use induction.

Your base case(s)? **Your code's base case(s).**

How do you break down the inductive step? **However your code breaks the problem down into smaller cases.**

Inductive hypothesis? **The recursive calls work for smaller-sized inputs.**

Proving a Recursive Algorithm is Correct

```
// Pre: n >= 0.  
// Post: returns n!  
int fact(int n) {  
    if (n == 0) return 1;
```

```
    else  
        return n*fact(n-1);
```

```
}
```

$$\underline{\text{Def.}} \quad n! = \begin{cases} 1 & n=0 \\ n \times (n-1)! & n>0 \end{cases}$$

Prove: fact(n) = n!

Base case: n = 0

fact(0) returns 1 and $0! = 1$
by definition.

Inductive hyp: fact(n)
returns n! for all $n \leq k$.

strong induction

Inductive step: For $n = k + 1$,
code returns n*fact(n-1).

By IH, fact(n-1) is $(n-1)!$
and $n! = n * (n-1)!$ by
definition.

by i.h. $(n-1)!$

Proving a Recursive Algorithm is Correct

Problem: Prove that our algorithm for randomly permuting a string gives an equal chance of returning every permutation (assuming `rand()` works as advertised).

Induction on the length of $S(n)$

Base case: strings of length 1 have only one permutation.

$\rightarrow \frac{1}{n!} = \text{prob. of each permutation}$

Induction hypothesis: Assume that our call to `permute(S, n-1)` works (randomly permutes the first $n-1$ characters of S).

We choose the last letter uniformly at random from the string. To get a random permutation, we need only randomly permute the remaining letters. `permute(S, n-1)` does exactly that.

exact proof

vague

requires conditional probabilities

Recurrence Relations... Already Covered

See Runtime Examples #5-7.

Additional Problem: Prove binary search takes $O(\lg n)$ time.

```
// Search A[i..j] for key.
// Return index of key or -1 if key not found.
int bSearch(int A[], int key, int i, int j) {
    if (j < i) return -1;
    int mid = (i + j) / 2;
    if (key < A[mid])
        return bSearch(A, key, i, mid-1);
    else if (key > A[mid])
        return bSearch(A, key, mid+1, j);
    else return mid;
}
```

Binary Search Problem (Worked)

Note: Let n be # of elements considered in the array, $n = j - i + 1$.

```
int bSearch(int A[], int key, int i, int j) {  
    if (j < i) return -1;  $O(1)$  base case  
    int mid = (i + j) / 2;  $O(1)$   
    if (key < A[mid])  $O(1)$   
        return bSearch(A, key, i, mid-1);  $T(\lfloor \frac{n-1}{2} \rfloor) \leq T(\lfloor n/2 \rfloor)$   
    else if (key > A[mid])  $O(1)$   
        return bSearch(A, key, mid+1, j);  $T(\lceil \frac{n-1}{2} \rceil) = T(\lfloor n/2 \rfloor)$   
    else return mid;  $O(1)$  }
```

Binary Search Problem (Worked)

The worst-case running time:

$$T(n) = \begin{cases} 1 & \text{if } n = 0 \\ T(\lfloor n/2 \rfloor) + 1 & \text{if } n > 0 \end{cases}$$

Guess:

$$T(n) = \lceil \lg(n + 1) \rceil + 1$$

$$T(0) = 1$$

$$T(1) = T(0) + 1 = 2$$

$$T(2) = T(1) + 1 = 3$$

$$T(3) = T(1) + 1 = 3$$

$$T(4) = T(2) + 1 = 4$$

$$T(5) = T(2) + 1 = 4$$

$$T(6) = T(3) + 1 = 4$$

$$T(7) = T(3) + 1 = 4$$

...

Binary Search Problem (Worked)

Claim $T(n) = \lceil \lg(n + 1) \rceil + 1$

Proof (by induction on n)

Base: $T(0) = 1 = \lceil \lg(0 + 1) \rceil + 1$

Ind.Hyp: $T(n) = \lceil \lg(n + 1) \rceil + 1$ for all $n \leq k$ (for some $k \geq 0$).

Ind.Step: For $n = k + 1$, $T(k + 1) = T(\lfloor \frac{k+1}{2} \rfloor) + 1 =$

If k is even	If k is odd
$= T(\frac{k}{2}) + 1$ (k is even)	$= T(\frac{k+1}{2}) + 1$ (k is odd)
$\stackrel{(IH)}{=} (\lceil \lg(\frac{k}{2} + 1) \rceil + 1) + 1$	$\stackrel{(IH)}{=} (\lceil \lg(\frac{k+1}{2} + 1) \rceil + 1) + 1$
$= \lceil \lg(2(\frac{k}{2} + 1)) \rceil + 1$	$= \lceil \lg(2(\frac{k+1}{2} + 1)) \rceil + 1$
$= \lceil \lg(k + 2) \rceil + 1$	$= \lceil \lg(k + 3) \rceil + 1$
	$= \lceil \lg(k + 2) \rceil + 1$ (k is odd)

Proving an Algorithm with Loops is Correct

Maybe we can use the same techniques we use for proving correctness of recursion to prove correctness of loops...

We do this by stating and proving “invariants”, properties that are always true (don’t vary) at particular points in the program.

One way of thinking of a loop is that it starts with a true invariant and does work to keep the invariant true for the next iteration of the loop.

Insertion Sort

```
void insertionSort(int A[], int length) {  
    for (int i = 1; i < length; i++) {  
        // Invariant: the elements in A[0..i-1] are in sorted order  
        int val = A[i];  
        int j;  
        for( j = i; j > 0 && A[j-1] > val; j-- )  
            A[j] = A[j-1];  
        A[j] = val;  
    }  
}
```

inserting A[i]
to correct
position

worst-case running time:

$$1 + 2 + \dots + n - 1 = \frac{(n-1)n}{2} \in \Theta(n^2)$$

Proving a Loop Invariant

Induction variable: Number of times through the loop.

(different from the C++ variables used in code!)

Base case: Prove the invariant true before the loop starts.

Induction hypothesis: Assume the invariant holds just before beginning some (unspecified) iteration.

Inductive step: Prove the invariant holds at the end of that iteration for the next iteration.

Extra bit: Make sure the loop will eventually end!

Loop termination: *check what loop invariant implicates when the loop is finished!*
We'll prove insertion sort works, but the cool part is not proving it works. The cool part is that the proof is a natural way to think about it working!

Insertion Sort

```
for (int i = 1; i < length; i++) {  
    // Invariant: the elements in A[0..i-1] are in sorted order  
    int val = A[i];  
    int j;  
    for( j = i; j > 0 && A[j-1] > val; j-- )  
        A[j] = A[j-1];  
    A[j] = val;  
}
```

Base case (at the start of the ($i = 1$) iteration): $A[0..0]$ only has one element; so, it's always in sorted order.

Insertion Sort

```
for (int i = 1; i < length; i++) {  
    // Invariant: the elements in A[0..i-1] are in sorted order  
    int val = A[i];  
    int j;  
    for( j = i; j > 0 && A[j-1] > val; j-- )  
        A[j] = A[j-1];  
    A[j] = val;  
}
```

Induction Hypothesis: At the start of iteration i of the loop, $A[0..i-1]$ are in sorted order.

Insertion Sort

```
for (int i = 1; i < length; i++) {  
    // Invariant: the elements in A[0..i-1] are in sorted order  
    int val = A[i];  
    int j;  
    for( j = i; j > 0 && A[j-1] > val; j-- )  
        A[j] = A[j-1];  
    A[j] = val;  
}
```

Inductive Step:

- ▶ The inner loop places $val = A[i]$ at the appropriate index $j < i$ by shifting elements of $A[0..i-1]$ that are larger than val one position to the right.
- ▶ Since $A[0..i-1]$ is sorted (by IH), $A[0..i]$ ends up in sorted order and the invariant holds at the start of the next iteration ($i = i + 1$).

Insertion Sort

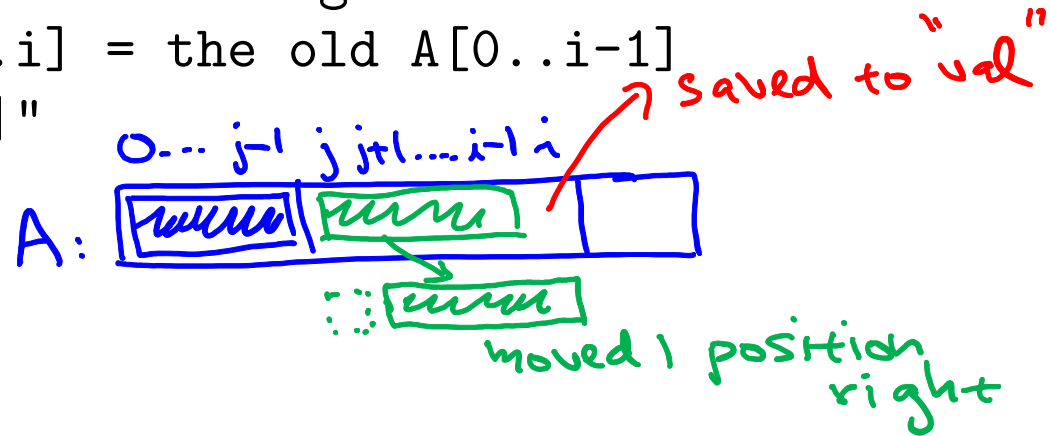
```
for (int i = 1; i < length; i++) {  
    // Invariant: the elements in A[0..i-1] are in sorted order  
    int val = A[i];  
    int j;  
    for( j = i; j > 0 && A[j-1] > val; j-- )  
        A[j] = A[j-1];  
    A[j] = val;  
}
```

Loop termination:

- ▶ The loop ends after $\text{length} - 1$ iterations.
- ▶ When it ends, we were about to enter the $(i = \text{length})$ iteration.
- ▶ Therefore, by the newly proven invariant, when the loop ends, $A[0..length-1]$ is in sorted order... which means A is sorted!

Practice: Prove the Inner Loop Correct

```
for (int i = 1; i < length; i++) {  
    // Invariant: the elements in A[0..i-1] are in sorted order  
    int val = A[i];  
    int j;  
    for( j = i; j > 0 && A[j-1] > val; j-- )  
        // What's the invariant? Something like  
        // "A[0..j-1] + A[j+1..i] = the old A[0..i-1]"  
        // and val < A[j+1..i]"  
        A[j] = A[j-1];  
    A[j] = val;  
}
```



Prove by induction that the inner loop operates correctly. (This may feel unrealistically easy!)

Finish the proof! (As we did for the outer loop, talk about what the invariant means when the loop ends.)

Recursion vs. Iteration

Which one is better? Recursion or iteration?

Simple
(easier to write)

more efficient
(because no call stack)

Simulating a Loop with Recursion

```
int i = 0
while (i < n)
    foo(i)
    i++
```

```
recFoo(0, n)
```

where recFoo is:

```
void recFoo(int i, int n) {
    if (i < n) {
        foo(i)
        recFoo(i + 1, n)
    }
}
```

Anything we can do with iteration, we can do with recursion.

Simulating Recursion with a Stack

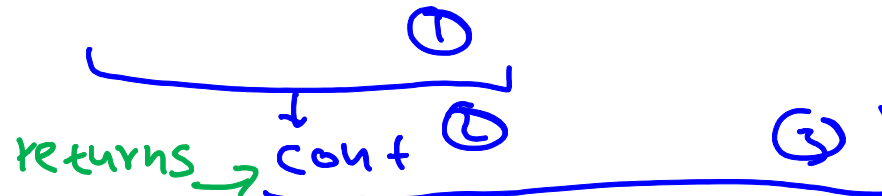
How does recursion work in a computer?

Each function call generates an *activation record*—holding local variables and the program point to return to—which is pushed on a stack (the *call stack*) that tracks the current chain of function calls.

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

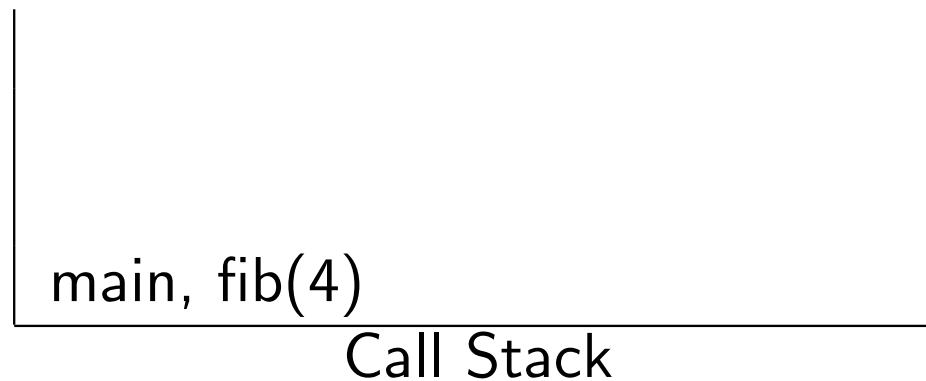
- ① call fib(4)
- ② print result
- ③ print newline

```
int main() { cout << fib(4) << endl; }
```



Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```



Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```

Line 1, n=4
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```

Line 2, n=4, a=fib(3)
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```

→ Line 1, n=3
Line 2, n=4, a=fib(3)
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```

Line 2, n=3, a=fib(2)

Line 2, n=4, a=fib(3)

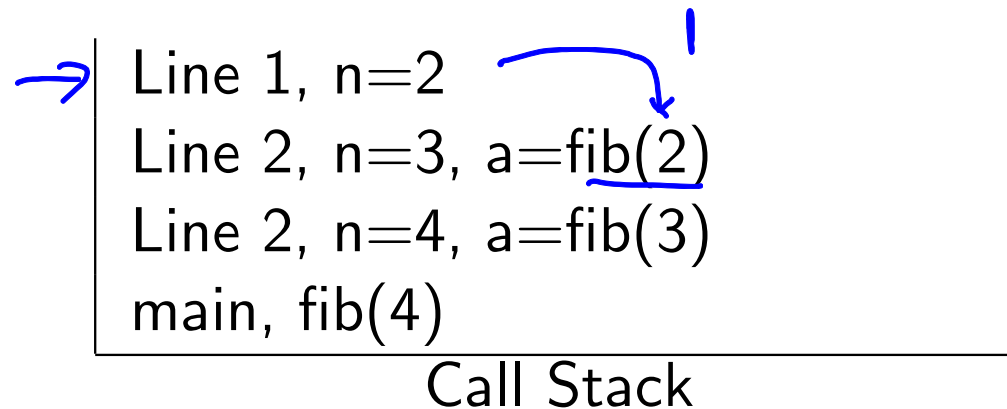
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```




Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```

Line 1, n=2, return 1
Line 2, n=3, a=fib(2)
Line 2, n=4, a=fib(3)
main, fib(4)



Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```

```
Line 2, n=3, a=1  
Line 2, n=4, a=fib(3)  
main, fib(4)
```

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.  if (n <= 2) return 1;  
2.  int a = fib(n-1);  
3.  int b = fib(n-2);  
4.  return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```

```
Line 3, n=3, a=1, b=fib(1)  
Line 2, n=4, a=fib(3)  
main, fib(4)
```

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```


Line 1, n=1
Line 3, n=3, a=1, b=fib(1)
Line 2, n=4, a=fib(3)
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```

Line 1, n=1, return 1 
Line 3, n=3, a=1, b=fib(1)
Line 2, n=4, a=fib(3)
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.  if (n <= 2) return 1;  
2.  int a = fib(n-1);  
3.  int b = fib(n-2);  
4.  return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```

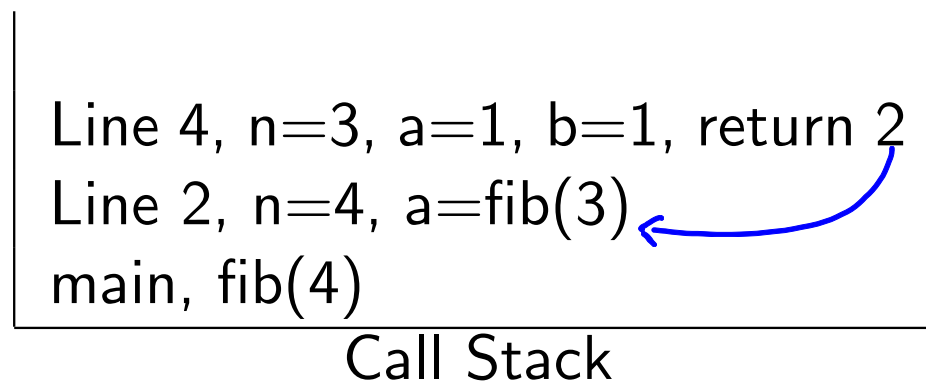
```
Line 3, n=3, a=1, b=1  
Line 2, n=4, a=fib(3)  
main, fib(4)
```

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```



Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```

Line 3, n=4, a=2, b=fib(2)
main, fib(4)

Call Stack

Simulating Recursion with a Stack


```
int fib(int n) {  
1.  if (n <= 2) return 1;  
2.  int a = fib(n-1);  
3.  int b = fib(n-2);  
4.  return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```

```
Line 1, n=2  
Line 3, n=4, a=2, b=fib(2)  
main, fib(4)
```

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```

Line 1, n=2, return 1 
Line 3, n=4, a=2, b=fib(2)
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```

Line 3, n=4, a=2, b=1
main, fib(4)


Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```

Line 4, n=4, a=2, b=1, return 3
main, fib(4)

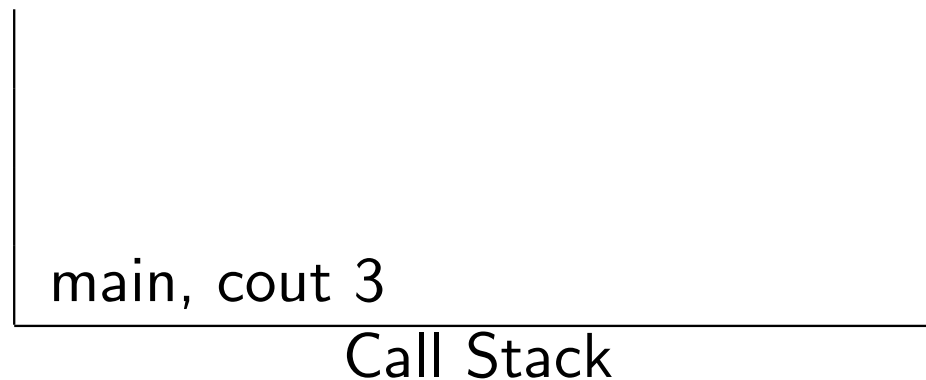


Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```



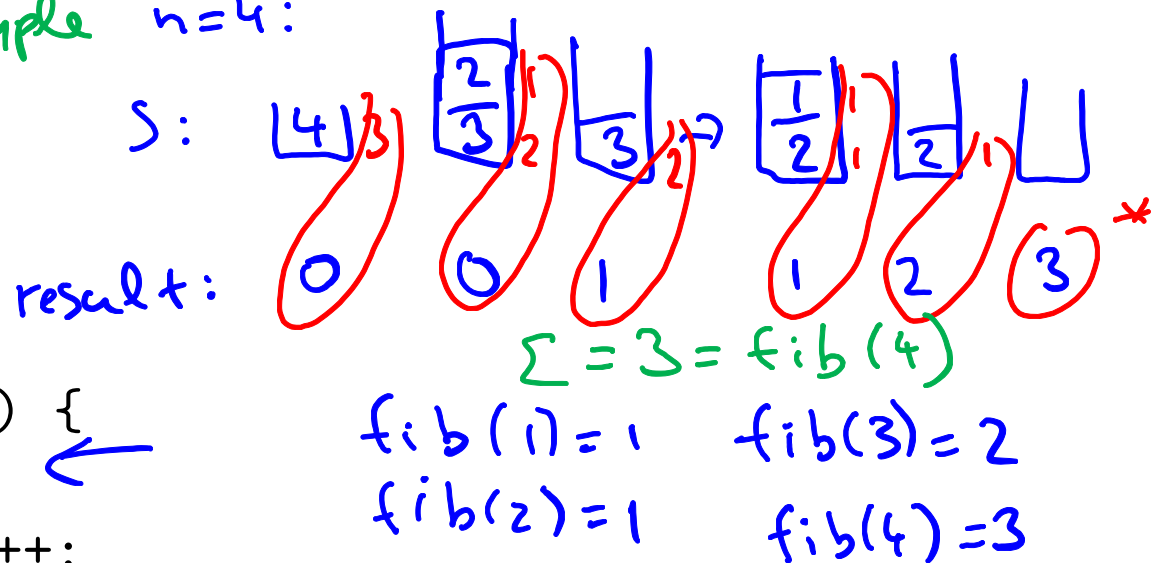
Simulating Recursion with a Stack

How do we simulate fib with a stack?

That's what our computer already does. We can sometimes do it a bit more efficiently by only storing what's really needed on the stack:

Example $n=4$:

```
int fib(int n) {
  int result = 0;
  Stack S;
  S.push(n);
  while( !S.is_empty() ) {
    int k = S.pop();
    if( k <= 2 ) result++;
    else { S.push(k - 1); S.push(k - 2); }
  }
  return result;
}
```



* red numbers : apply fib on the # in stack

Loop Invariant for Correctness

```
int fib(int n) {
    int result = 0;
    Stack S;
    S.push(n);
    while( !S.is_empty() ) {
        // Invariant: ??
        int k = S.pop();
        if( k <= 2 ) result++;
        else { S.push(k - 1); S.push(k - 2); }
    }
    return result;
}
```

What is the loop invariant?

Loop Invariant for Correctness

```
int fib(int n) {
    int result = 0;
    Stack S;
    S.push(n);
    while( !S.is_empty() ) {
        // Invariant:  $\longrightarrow \left( \text{result} + \sum_{k \text{ on Stack}} \text{fib}_k \right) = \underline{\underline{\text{fib}_n}}$ 
        int k = S.pop();
        if( k <= 2 ) result++;
        else { S.push(k - 1); S.push(k - 2); }
    }
    return result;
}
```

Prove Invariant using induction.

Base (zero iterations): $\left(\underline{\text{result}} + \sum_{k \text{ on Stack}} \text{fib}_k \right) = 0 + \text{fib}_n = \underline{\underline{\text{fib}_n}}$ ✓

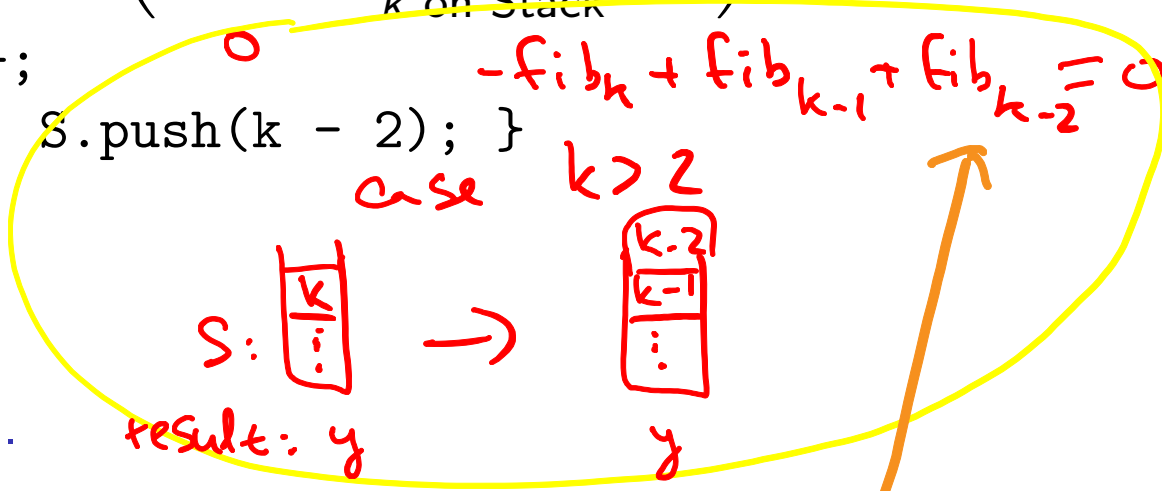
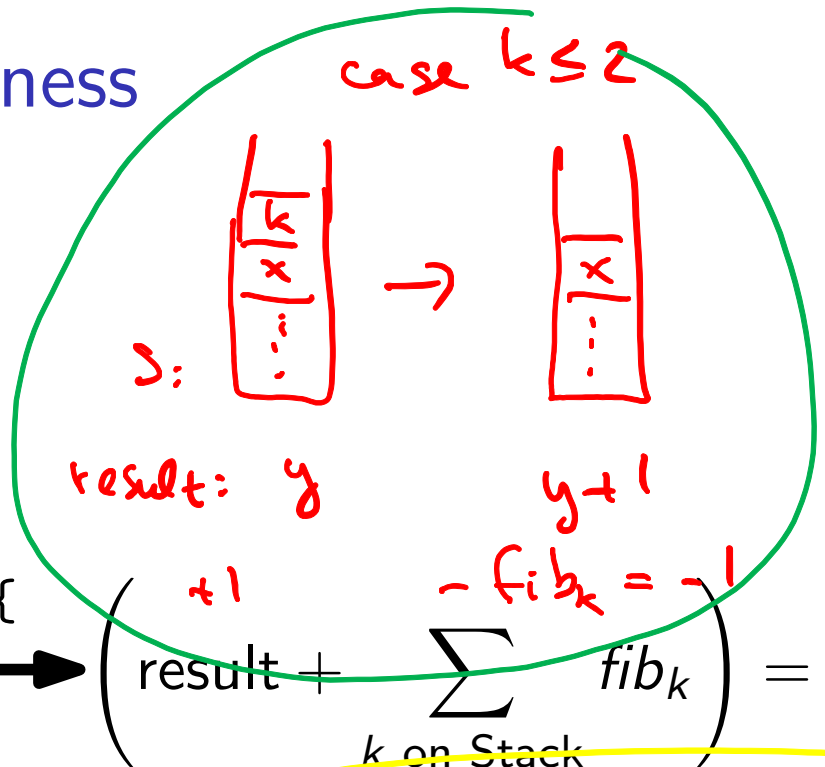
Handwritten notes: S: [n], result: 0

Loop Invariant for Correctness

```

int fib(int n) {
  int result = 0;
  Stack S;
  S.push(n);
  while( !S.is_empty() ) {
    // Invariant:
    int k = S.pop();
    if( k <= 2 ) result++;
    else { S.push(k - 1); S.push(k - 2); }
  }
  return result;
}

```



Prove Invariant using induction.

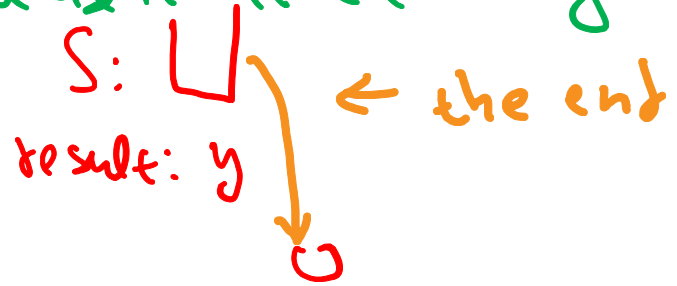
Ind. Step: If $k \leq 2$ then result increases by $fib_k = 1$ and k is removed from stack. Invariant preserved. If $k > 2$ then k is replaced by $k - 1$ and $k - 2$ on stack. Since $fib_k = fib_{k-1} + fib_{k-2}$, invariant preserved.

def. of fib. numbers

Loop Invariant for Correctness

Note: on the previous slides we proved invariant is correct, now we will use it at the end of the loop:

```
int fib(int n) {
    int result = 0;
    Stack S;
    S.push(n);
    while( !S.is_empty() ) {
        // Invariant:
        int k = S.pop();
        if( k <= 2 ) result++; *
        else { S.push(k - 1); S.push(k - 2); }
    }
    return result;
}
```



Invariant: $\left(\text{result} + \sum_{k \text{ on Stack}} \text{fib}_k \right) = \underline{\text{fib}_n}$

result++; * \Rightarrow result = fib_n at the end

executes $\text{fib}(n)$ times

Prove Invariant using induction.

End: When loop terminates, stack is empty so result = fib_n .

Q. running time? A.

$\in \Theta(\varphi^n)$

so runtime is in $\Omega(\varphi^n)$

Recursion vs. Iteration

Depends ...

Which one is more elegant? Recursion or iteration?

↓
Merge Sort
Tower of Hanoi

↓
Linear Search

Recursion vs. Iteration

Which one is more efficient? Recursion or iteration?

no need for call stack

— constant times
less time

— less space

Accidentally Making Lots of Recursive Calls; Recall...

Recursive Fibonacci:

```
int fib(int n) {  
    if (n <= 2) return 1;  
    else      return fib(n-1) + fib(n-2);  
}
```

Lower bound analysis

$$T(n) \geq \begin{cases} b & \text{if } n = 0, 1 \\ T(n-1) + T(n-2) + c & \text{if } n > 1 \end{cases}$$

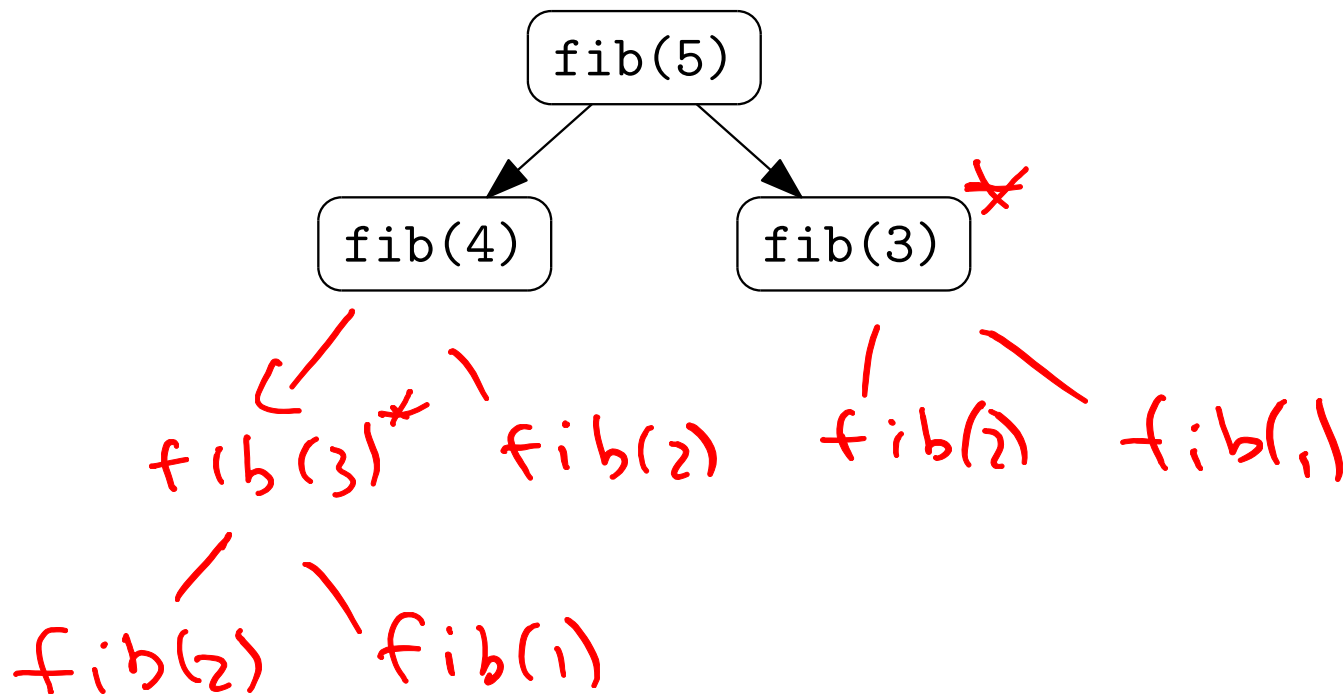
$$T(n) \geq b\varphi^{n-1} \quad \in \Theta(\varphi^n)$$

where $\varphi = (1 + \sqrt{5})/2$.

Accidentally Making Lots of Recursive Calls; Recall...

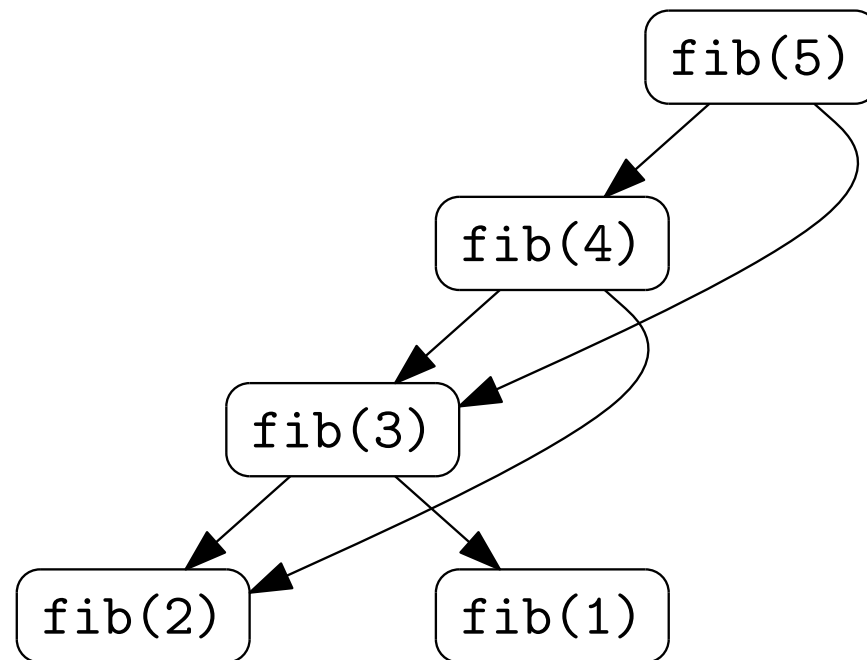
```
int fib(int n) {  
    if (n <= 2) return 1;  
    else      return fib(n-1) + fib(n-2);  
}
```

Finish the recursion tree for fib(5)...



Fixing Fib with Iteration

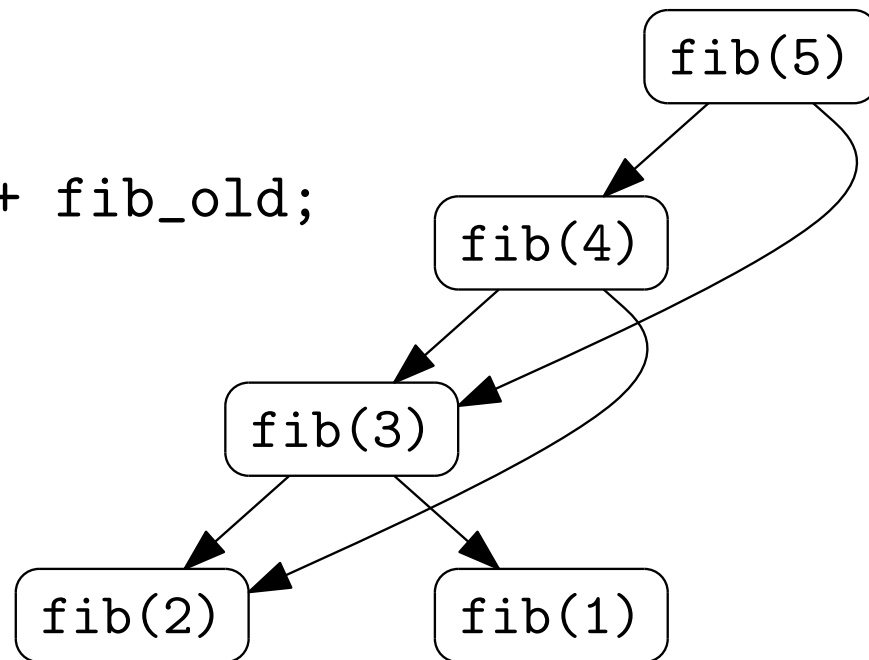
What we really want is to “share” nodes in the recursion tree:



Fixing Fib with Iteration and “Dynamic Programming”

Here’s one fix that “walks up” the left of the tree:

```
int fib_dp(int n) {  
    int fib_old = 1;  
    int fib = 1;  
    int fib_new;  
    while (n > 2) {  
        int fib_new = fib + fib_old;  
        fib_old = fib;  
        fib = fib_new;  
        --n;  
    }  
    return fib;  
}
```



Fixing Fib with Recursion and “Memoizing”

Here’s another fix that stores solutions it has calculated before:

```
int* fib_solns = new int[big_enough](); // init to 0

fib_solns[1] = 1;

fib_solns[2] = 1;

int fib_memo(int n) {
    // If we don't know the answer, compute it.
    if (fib_solns[n] == 0)
        fib_solns[n] = fib_memo(n-1) + fib_memo(n-2);
    return fib_solns[n];
}
```

Recursion vs. Iteration

Which one is more efficient? Recursion or iteration?

It's probably easier to shoot yourself in the foot without noticing when you use recursion, and the call stack may carry around more memory than you really need to store, but otherwise...

Neither is more efficient (asymptotically).

Managing the Call Stack: Tail Recursion

```
void endlesslyGreet() {  
    cout << "Hello, world!" << endl;  
    endlesslyGreet();  
}
```

This is clearly infinite recursion. The call stack will get as deep as it can get and then bomb, right?

But... why have a call stack? There's no (need to) return to the caller.

Try compiling it with at least `-O2` optimization and running. It won't give a stack overflow!

Tail Recursion

A function is “tail recursive” if for any recursive call in the function, that call is the last thing the function needs to do before returning.

In that case, why bother pushing a new activation record? There's no reason to return to the caller. Just use the current record.

That's what most compilers will do.

Tail Recursive?

No

```
int fib(int n) {  
    if (n <= 2) return 1; ① ⑤ ②  
    else return fib(n-1) + fib(n-2);  
}
```

neither recursive call is tail call

Tail Recursive?

NO

```
int fact(int n) {  
    if (n == 0) return 1;  
    else      return n * fact(n - 1);  
}
```

after call returns

Tail Recursive?

YES

```
int fact(int n) { return fact_acc(n, 1); }
```

```
int fact_acc(int b, int acc) {  
    if (b == 0) return acc;  
    else      return fact_acc(b - 1, acc * b);  
}
```

invariant:

$$n! = acc * b!$$

$$(b+1) * (b+2) * \dots * n$$

Exercise:

Prove the invariant
using induction.

```
int fact(n) {  
    int acc = 1;  
    int b = n;  
    while (b > 0) {  
        acc = acc * b;  
        b = b - 1;  
    }  
    return acc;  
}
```

Prove Invariant:

$$acc \times b! = n!$$

Base: $b = n, acc = 1$ (first call to fact_acc)

$$acc \times b! = 1 \times n! = n! \quad \checkmark$$

Inductive step:

ind. hyp. :

new values:

($(l+1)$ -st call)

$$acc \times b! = n! \quad (\text{l'st call})$$

$$b_{\text{new}} = b - 1 \quad acc_{\text{new}} = acc \times b$$

$$\begin{aligned} acc_{\text{new}} \times b_{\text{new}}! &= acc \times b \times \underbrace{(b-1)!}_{b!} \\ &= acc \times b! \stackrel{\text{i.h.}}{=} n! \end{aligned}$$

End of induction.

Termination ($b = 0$): $acc \times 0! = n!$

$$\underline{acc = n!} \leftarrow \text{return value} \quad \checkmark$$

Side Note: Tail Calls

```
int fact(int n) { return fact_acc(n, 1); }
```

```
int fact_acc(int b, int acc) {  
    if (b == 0) return acc;  
    else      return fac_acc(b - 1, acc * b);  
}
```

Actually we can talk about any function call being a “tail call”, even if it’s not recursive. E.g., the call to `fact_acc` in `fact` is a tail call: no need to extend the stack.

Eliminating Tail Recursion

```
// Search A[i..j] for key.
// Return index of key or -1 if key not found.
int bSearch(int A[], int key, int i, int j) {
    while (j >= i) {
        int mid = (i + j) / 2;
        if (key < A[mid])
            j = mid - 1;
        else if (key > A[mid])
            i = mid + 1;
        else return mid;
    }
    return -1;
}
```

both tail calls

just update the variables instead of making recursive calls