University of British Columbia
CPSC 314 Computer Graphics
Jan-Apr 2016

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Math Basics

Week 1, Fri Jan 8

http://www.ugrad.cs.ubc.ca/~cs314/Vjan2016

Readings For Lecture

- Shirley/Marschner (3rd edition)
  - Ch 2: Miscellaneous Math, Sec 2.1-2.4
  - Ch 5: Linear Algebra, Sec 5.1-5.3
- Gortler
  - Ch 2: Linear, Sec 2.1 – 2.4

Vectors and Matrices

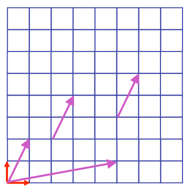
Notation: Scalars, Vectors, Matrices

- scalar a
  - (lower case, italic)
- vector $\mathbf{a} = [a_1 \quad a_2 \quad \dots \quad a_n]$
  - (lower case, bold)
- matrix
  - (upper case, bold)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Vectors

- arrow: length and direction
  - oriented segment in nD space
- offset / displacement
  - location if given origin



Column vs. Row Vectors

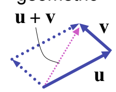
- row vectors $\mathbf{a}_{row} = [a_1 \quad a_2 \quad \dots \quad a_n]$
- column vectors $\mathbf{a}_{col} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix}$
- switch back and forth with transpose

$$\mathbf{a}_{col}^T = \mathbf{a}_{row}$$

Vector-Vector Addition

- add: vector + vector = vector
- parallelogram rule
  - tail to head, complete the triangle

geometric



algebraic

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$$

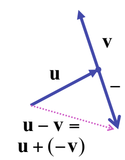
examples: $(3,2) + (6,4) = (9,6)$

$(2,5,1) + (3,1,-1) = (5,6,0)$

Vector-Vector Subtraction

- subtract: vector - vector = vector

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{bmatrix}$$



$(3,2) - (6,4) = (-3,-2)$

$(2,5,1) - (3,1,-1) = (-1,4,2)$

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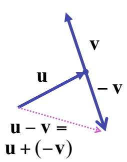

$(3,2) - (6,4) = (-3,-2)$

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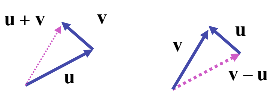
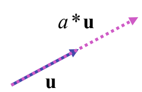
argument reversal



Scalar-Vector Multiplication

- multiply: scalar * vector = vector
  - vector is scaled



$$a * \mathbf{u} = (a * u_1, a * u_2, a * u_3)$$

$2 * (3,2) = (6,4)$

$.5 * (2,5,1) = (1,2.5,.5)$

Vector-Vector Multiplication: Dot

- multiply v1: vector * vector = scalar
- dot product, aka inner product $\mathbf{u} \bullet \mathbf{v}$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (u_1 * v_1) + (u_2 * v_2) + (u_3 * v_3)$$

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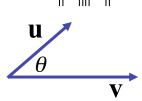
$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

- geometric interpretation
  - lengths, angles
  - can find angle between two vectors

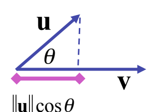


Dot Product Geometry

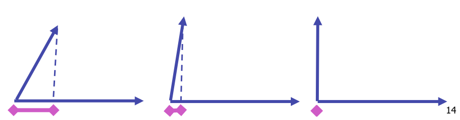
- can find length of projection of u onto v

$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

$$\|\mathbf{u}\|\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|}$$



- as lines become perpendicular, $\mathbf{u} \bullet \mathbf{v} \to 0$



Dot Product Example

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (u_1 * v_1) + (u_2 * v_2) + (u_3 * v_3)$$

$$\begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix} = (6*1) + (1*7) + (2*3) = 6 + 7 + 6 = 19$$

Vector-Vector Multiplication, Cross

- multiply v2: vector * vector = vector
- cross product
  - algebraic

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Vector-Vector Multiplication, Cross

- multiply v2: vector * vector = vector
- cross product
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1

$$\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{bmatrix}$$

2

$$\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{bmatrix}$$

3

$$\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{bmatrix}$$

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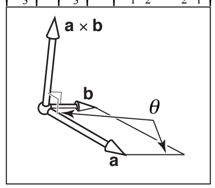
  - geometric

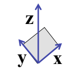
$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$$

- $\|\mathbf{a}\times\mathbf{b}\|$ parallelogram area
- $\mathbf{a}\times\mathbf{b}$ perpendicular to parallelogram



RHS vs. LHS Coordinate Systems

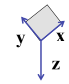
- right-handed coordinate system convention



right hand rule:
index finger x, second finger y;
right thumb points up

$$\mathbf{z} = \mathbf{x}\times\mathbf{y}$$

- left-handed coordinate system



left hand rule:
index finger x, second finger y;
left thumb points down

$$\mathbf{z} = \mathbf{x}\times\mathbf{y}$$

Matrix-Matrix Addition

- add: matrix + matrix = matrix

$$\begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix} + \begin{bmatrix}n_{11} & n_{12}\\ n_{21} & n_{22}\end{bmatrix} = \begin{bmatrix}n_{11}+m_{11} & n_{12}+m_{12}\\ n_{21}+m_{21} & n_{22}+m_{22}\end{bmatrix}$$

- example

$$\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix} + \begin{bmatrix}-2 & 5\\ 7 & 1\end{bmatrix} = \begin{bmatrix}1+(-2) & 3+5\\ 2+7 & 4+1\end{bmatrix} = \begin{bmatrix}-1 & 8\\ 9 & 5\end{bmatrix}$$

Scalar-Matrix Multiplication

- multiply: scalar * matrix = matrix

$$a\begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix} = \begin{bmatrix}a*m_{11} & a*m_{12}\\ a*m_{21} & a*m_{22}\end{bmatrix}$$

- example

$$3\begin{bmatrix}2 & 4\\ 1 & 5\end{bmatrix} = \begin{bmatrix}3*2 & 3*4\\ 3*1 & 3*5\end{bmatrix} = \begin{bmatrix}6 & 12\\ 3 & 15\end{bmatrix}$$

Matrix-Matrix Multiplication

- can only multiply (n,k) by (k,m): number of left cols = number of right rows
  - legal

$$\begin{bmatrix}a & b & c\\ e & f & g\end{bmatrix}\begin{bmatrix}h & i\\ j & k\\ l & m\end{bmatrix}$$

  - undefined

$$\begin{bmatrix}a & b & c\\ e & f & g\\ o & p & q\end{bmatrix}\begin{bmatrix}h & i\\ j & k\end{bmatrix}$$

Matrix-Matrix Multiplication

- row by column

$$\begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}\begin{bmatrix}n_{11} & n_{12}\\ n_{21} & n_{22}\end{bmatrix} = \begin{bmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix}$$

$$p_{11} = m_{11}n_{11} + m_{12}n_{21}$$

Matrix-Matrix Multiplication

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- noncommutative: **AB != BA**

Matrix-Vector Multiplication

- points as column vectors: postmultiply

$$\begin{bmatrix}x'\\y'\\z'\\h'\end{bmatrix} = \begin{bmatrix}m_{11} & m_{12} & m_{13} & m_{14}\\ m_{21} & m_{22} & m_{23} & m_{24}\\ m_{31} & m_{32} & m_{33} & m_{34}\\ m_{41} & m_{42} & m_{43} & m_{44}\end{bmatrix}\begin{bmatrix}x\\y\\z\\h\end{bmatrix} \qquad \mathbf{p'} = \mathbf{Mp}$$

- points as row vectors: premultiply

$$\begin{bmatrix}x' & y' & z' & h'\end{bmatrix} = \begin{bmatrix}x & y & z & h\end{bmatrix}\begin{bmatrix}m_{11} & m_{12} & m_{13} & m_{14}\\ m_{21} & m_{22} & m_{23} & m_{24}\\ m_{31} & m_{32} & m_{33} & m_{34}\\ m_{41} & m_{42} & m_{43} & m_{44}\end{bmatrix}^T \qquad \mathbf{p'}^T = \mathbf{p}^T\mathbf{M}^T$$

Matrices

- transpose

$$\begin{bmatrix}m_{11} & m_{12} & m_{13} & m_{14}\\ m_{21} & m_{22} & m_{23} & m_{24}\\ m_{31} & m_{32} & m_{33} & m_{34}\\ m_{41} & m_{42} & m_{43} & m_{44}\end{bmatrix}^T = \begin{bmatrix}m_{11} & m_{21} & m_{31} & m_{41}\\ m_{12} & m_{22} & m_{32} & m_{42}\\ m_{13} & m_{23} & m_{33} & m_{43}\\ m_{14} & m_{24} & m_{34} & m_{44}\end{bmatrix}$$

- identity

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

- inverse $\mathbf{AA}^{-1} = \mathbf{I}$
  - not all matrices are invertible

Matrices and Linear Systems

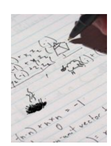
- linear system of n equations, n unknowns

$$3x + 7y + 2z = 4$$
$$2x - 4y - 3z = -1$$
$$5x + 2y + z = 1$$

- matrix form **Ax=b**

$$\begin{bmatrix}3 & 7 & 2\\ 2 & -4 & -3\\ 5 & 2 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}4\\-1\\1\end{bmatrix}$$

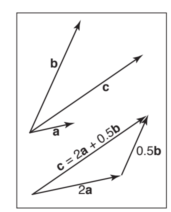
Basis Vectors and Frames



Basis Vectors

- take any two vectors that are linearly independent (nonzero and nonparallel)
  - can use linear combination of these to define any other vector:

$$\mathbf{c} = w_1\mathbf{a} + w_2\mathbf{b}$$



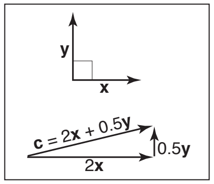
Orthonormal Basis Vectors

- if basis vectors are orthonormal: orthogonal (mutually perpendicular) and unit length
  - we have Cartesian coordinate system
  - familiar Pythagorean definition of distance

orthonormal algebraic properties

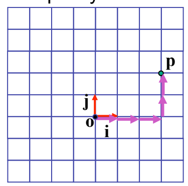
$\|\mathbf{x}\| = \|\mathbf{y}\| = 1,$

$\mathbf{x} \bullet \mathbf{y} = 0$



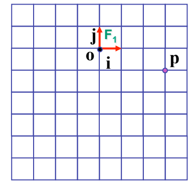
Basis Vectors and Origins

- coordinate system: just basis vectors
  - can only specify offset: vectors
- coordinate frame: basis vectors and origin
  - can specify location as well as offset: points



$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$

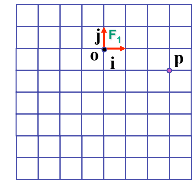
Working with Frames



$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$

F_1

Working with Frames



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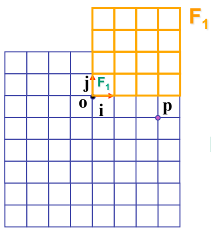
F_1 **p = (3,-1)**

Working with Frames



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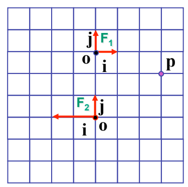
Working with Frames

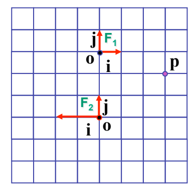


$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$

F_1 **p = (3,-1)**

F_2

Working with Frames

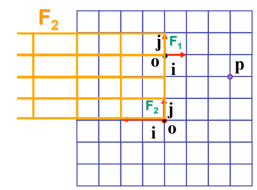


$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$

F_1 **p = (3,-1)**

F_2 **p = (-1.5,2)**

Working with Frames



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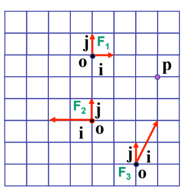
Working with Frames



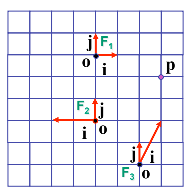
$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$

F_1 **p = (3,-1)**

F_2 **p = (-1.5,2)**

F_3

Working with Frames



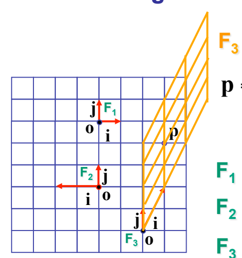
$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$

F_1 **p = (3,-1)**

F_2 **p = (-1.5,2)**

F_3 **p = (1,2)**

Working with Frames



$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$

F_1 **p = (3,-1)**

F_2 **p = (-1.5,2)**

F_3 **p = (1,2)**

Named Coordinate Frames

- origin and basis vectors $\mathbf{p} = \mathbf{o} + a\mathbf{x} + b\mathbf{y} + c\mathbf{z}$
- pick canonical frame of reference
  - then don't have to store origin, basis vectors
  - just $\mathbf{p} = (a, b, c)$
  - convention: Cartesian orthonormal one on previous slide
- handy to specify others as needed
  - airplane nose, looking over your shoulder, ...
  - really common ones given names in CG
    - object, world, camera, screen, ...